Ficha 4

$\boxed{1}$

a) V é um espaço vetorial se lhe estão associados duas operações, uma adição de elementos de V e uma multiplicação de números reais por elementos de V, com os seguintes propriedades: $(V = \mathbb{R}^2)$

Fecho da adição → $\boxed{1}$ A soma de um par de elementos de V pertence a V

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix}$$

$$\begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix} \in V ?$$

$$\begin{matrix} x_1+y_1-1 \in \mathbb{R} \\ x_2+y_2+1 \in \mathbb{R} \end{matrix} \quad , \text{ logo } \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix} \in \mathbb{R}^2 \Leftrightarrow \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix} \in V \checkmark$$

Fecho da multiplicação por números reais → $\boxed{2}$ O produto de qualquer número real por qualquer elemento de V pertence a V

$$\alpha \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 - \alpha + 1 \\ \alpha x_2 + \alpha - 1 \end{bmatrix}, \quad \alpha \in \mathbb{R}$$

$$\left(\begin{bmatrix} \alpha x_1 - \alpha + 1 \\ \alpha x_2 + \alpha - 1 \end{bmatrix}, \ \alpha \in \mathbb{R} \right) \in V ?$$

$$\begin{matrix} (\alpha x_1 - \alpha + 1, \alpha \in \mathbb{R}) \in \mathbb{R} \\ (\alpha x_2 + \alpha - 1, \alpha \in \mathbb{R}) \in \mathbb{R} \end{matrix} \quad , \text{ logo } \begin{bmatrix} \alpha x_1 - \alpha + 1 \\ \alpha x_2 + \alpha - 1 \end{bmatrix} \in \mathbb{R}^2 \Leftrightarrow \begin{bmatrix} \alpha x_1 - \alpha + 1 \\ \alpha x_2 + \alpha - 1 \end{bmatrix} \in V \checkmark$$

Comutatividade da adição → $\boxed{3}$ $x \oplus y = y \oplus x, \ \forall x, y \in V$

$$x \oplus y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix} = \begin{bmatrix} y_1+x_1-1 \\ y_2+x_2+1 \end{bmatrix} = y \oplus x \checkmark$$

Associatividade da adição → $\boxed{4}$ $(x \oplus y) \oplus z = x \oplus (y \oplus z), \ \forall x, y, z \in V$

$$x \oplus y = \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix}$$

$$(x \oplus y) \oplus z = \begin{bmatrix} x_1+y_1-1 \\ x_2+y_2+1 \end{bmatrix} \oplus \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (x_1+y_1-1)+z_1-1 \\ (x_2+y_2+1)+z_2+1 \end{bmatrix} = \begin{bmatrix} x_1+(y_1+z_1-1)-1 \\ x_2+(y_2+z_2+1)+1 \end{bmatrix}$$

$$= x \oplus (y \oplus z) \checkmark$$

Existência de zero → $\boxed{5}$ Existe um elemento V, designado por 0_V (zero), tal que $x + 0_V = x$, $\forall x \in V$

$$x \oplus 0_V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} 0_{V_1} \\ 0_{V_2} \end{bmatrix} = \begin{bmatrix} x_1 + 0_{V_1} - 1 \\ x_2 + 0_{V_2} + 1 \end{bmatrix}$$

$$x_1 + 0_{V_1} - 1 = x_1 \wedge x_2 + 0_{V_2} + 1 = x_2$$
$$\Leftrightarrow 0_{V_1} - 1 = 0 \wedge 0_{V_2} + 1 = 0$$
$$\Leftrightarrow 0_{V_1} = 1 \wedge 0_{V_2} = -1 \implies 0_V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x \oplus 0_V = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 + 1 - 1 \\ x_2 + (-1) + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x \checkmark$$

Existência de simétricos → $\boxed{6}$ Qualquer que seja o elemento x de V existe um elemento y de V a que se chama o simétrico de x: $x + y = 0_V$

$$x \oplus y = \begin{bmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 + 1 \end{bmatrix} \qquad 0_V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x \oplus y = 0_V \Leftrightarrow x_1 + y_1 - 1 = 1 \wedge x_2 + y_2 + 1 = -1$$
$$\Leftrightarrow y_1 = 2 - x_1 \wedge y_2 = -2 - x_2$$

$$y = \begin{bmatrix} 2 - x_1 \\ -2 - x_2 \end{bmatrix}$$

$$x \oplus y = \begin{bmatrix} \cancel{x_1} + (2 - \cancel{x_1}) - 1 \\ \cancel{x_2} + (-2 - \cancel{x_2}) + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0_V \checkmark$$

$\boxed{7}$ $\alpha \odot (\beta \odot x) = (\alpha\beta) \odot x$, $\forall a, b \in \mathbb{R}, \forall x \in V$

$$\beta \odot x = \begin{bmatrix} \beta x_1 - \beta + 1 \\ \beta x_2 + \beta - 1 \end{bmatrix} \qquad \alpha\beta x_1 - \alpha\beta + 1$$

$$\alpha \odot (\beta \odot x) = \begin{bmatrix} \alpha(\beta x_1 - \beta + 1) - \alpha + 1 \\ \alpha(\beta x_2 + \beta - 1) + \alpha - 1 \end{bmatrix} = \begin{bmatrix} \alpha\beta x_1 - \alpha\beta + \cancel{\alpha} - \cancel{\alpha} + 1 \\ \alpha\beta x_2 + \alpha\beta - \cancel{\alpha} + \cancel{\alpha} - 1 \end{bmatrix} = \begin{bmatrix} \alpha\beta x_1 - \alpha\beta + 1 \\ \alpha\beta x_2 + \alpha\beta - 1 \end{bmatrix}$$

$$= (\alpha\beta) \odot x \checkmark$$

$\boxed{8}$ $\alpha \odot (x \oplus y) = \alpha \odot x \oplus \alpha \odot y, \ \forall \alpha \in \mathbb{R}, \forall x, y \in V$

$$x \oplus y = \begin{bmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 + 1 \end{bmatrix}$$

$$\alpha \odot (x \oplus y) = \begin{bmatrix} \alpha(x_1 + y_1 - 1) - \alpha + 1 \\ \alpha(x_2 + y_2 + 1) + \alpha - 1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \alpha y_1 - 2\alpha + 1 \\ \alpha x_2 + \alpha y_2 + 2\alpha - 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha x_1 - \alpha + 1) + (\alpha y_1 - \alpha + 1) - 1 \\ (\alpha x_2 + \alpha - 1) + (\alpha y_2 + \alpha - 1) + 1 \end{bmatrix}$$

$$= \alpha \odot x \oplus \alpha \odot y \ \checkmark$$

$\boxed{9}$ $(\alpha + \beta) \odot x = \alpha \odot x + \beta \odot x, \ \forall \alpha, \beta \in \mathbb{R}, \forall x \in V$

$$(\alpha + \beta) \odot x = \begin{bmatrix} (\alpha+\beta) x_1 - (\alpha+\beta) + 1 \\ (\alpha+\beta) x_2 + (\alpha+\beta) - 1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta x_1 - \alpha - \beta + 1 \\ \alpha x_2 + \beta x_2 + \alpha + \beta - 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha x_1 - \alpha + 1) + (\beta x_1 - \beta + 1) - 1 \\ (\alpha x_2 + \alpha - 1) + (\beta x_2 + \beta - 1) + 1 \end{bmatrix}$$

$$= \alpha \odot x \oplus \beta \odot x \ \checkmark$$

$\boxed{10}$ $1 \odot x = x, \ \forall x \in V$

$$1 \odot x = \begin{bmatrix} 1 \times x_1 - \cancel{1} + \cancel{1} \\ 1 \times x_2 + \cancel{1} - \cancel{1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x \ \checkmark$$

Logo, V é um espaço vetorial.

$$0_V = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad e \quad \ominus x = \begin{bmatrix} 2 - x_1 \\ -2 - x_2 \end{bmatrix}$$

b)

$$S = \left\{ \begin{bmatrix} 1-2t \\ t-1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

① $0_V \in S$?

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in S$, sim! Quando $t=0$.

②

$\forall\, x, x' \in S$

$x \oplus x' \in S$?

$x \in S \Leftrightarrow x = \begin{bmatrix} 1-2z \\ z-1 \end{bmatrix}, z \in \mathbb{R}$

$x' \in S \Leftrightarrow x' = \begin{bmatrix} 1-2z' \\ z'-1 \end{bmatrix}, z' \in \mathbb{R}$

$x \oplus x' = \begin{bmatrix} 1-2z \\ z-1 \end{bmatrix} \oplus \begin{bmatrix} 1-2z' \\ z'-1 \end{bmatrix} = \begin{bmatrix} -2z-2z'+1 \\ z+z'-1 \end{bmatrix} = \Big[\, 1-2(z+z')$

③

$\forall\, \alpha \in \mathbb{R}, \forall\, x \in V$

$\alpha \odot x \in S$?

$x \in$

$\alpha \odot x = \alpha \odot \begin{bmatrix} 1-2z \\ z-1 \end{bmatrix} = \begin{bmatrix} \alpha(1-2z)-\alpha+1 \\ \alpha(z-1)+\alpha-1 \end{bmatrix} = \begin{bmatrix} 1-2\alpha z \\ \alpha z-1 \end{bmatrix}$

$t = \alpha z$, logo como $\alpha z \in \mathbb{R}$ logo $t \in \mathbb{R} \Rightarrow \alpha \odot x \in S$

$\boxed{2}$

a) $V = \mathbb{R}^2$, $S_i = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$, $S_{ii} = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (1,1)\}$

(i)

① $0_V \in S_i \Leftrightarrow (0,0) \in S_i$?

$x = 0 \wedge y = 0 \Rightarrow 0 + 0 = 0 \Leftrightarrow 0 = 0$, logo $0_V \in S_i$ ✓

② $\forall (x,y), (x',y') \in S_i$

$(x,y) + (x',y') \in S_i$?

$(x,y) \in S_i \Leftrightarrow \underline{x + y = 0}$

$(x',y') \in S_i \Leftrightarrow \underline{x' + y' = 0}$

$(x + x', y + y') \in S_i \Leftrightarrow x + x' + y + y' = 0 \Leftrightarrow \underbrace{x + y}_{0} + \underbrace{x' + y'}_{= 0} = 0 \Leftrightarrow 0 = 0$ ✓

③ $\forall \alpha \in \mathbb{R}$, $\forall (x,y) \in S_i$

$\alpha (x,y) \in S_i$?

$(x,y) \in S_i \Leftrightarrow \underline{x + y = 0}$

$(\alpha x, \alpha y) \in S_i \Leftrightarrow \alpha x + \alpha y = 0 \Leftrightarrow \alpha (x + y) = 0 \Leftrightarrow 0 = 0$ ✓

Logo, S_i é um subespaço vetorial de V

(ii)

① $0_V \in S_{ii} \Leftrightarrow (0,0) \in S_{ii}$?

$(0,0) \neq (1,1)$, logo $0_V \in S_{ii}$ ✓

② $\forall (x,y), (x',y') \in S_{ii}$

$(x,y) + (x',y') \in S_{ii}$?

Não, por exemplo para $(x,y) = (0,1) \in S_{ii}$ e $(x',y') = (1,0)$
$\Rightarrow (x + x', y + y') = (1,1) \notin S_{ii}$ ✗

Logo, S_{ii} não é um subespaço de V

b) $V = \mathbb{R}^3$, $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$

$0_V = (0,0,0)$

$0_V \in S$?

$x=0 \wedge y=0 \wedge z=0 \Rightarrow 0^2+0^2+0^2=1 \Leftrightarrow \underbrace{0=1}_{C.I.}$ ✗

Logo, S não é um subespaço de V

c)

(i) $\mathcal{P}_2 = \{\alpha_2 x^2 + \alpha_1 x + \alpha_0 , \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}\}$

$S = \{ax^2+bx+c \in \mathcal{P}_2 : c=0\} = ax^2+bx, \forall a,b \in \mathbb{R}$

$0_V = 0$

$0_V \in S$? Sim, para $x=0 \Rightarrow 0+0=0 \Leftrightarrow 0=0$

$\forall\, ax^2+bx+c, a'x^2+b'x^2+c' \in S$

$ax^2+bx+c+a'x^2+b'x^2+c' \in S$?

$ax^2+bx+c \in S \Rightarrow c=0$

$a'x^2+b'x+c' \in S \Rightarrow c'=0$

$ax^2+bx+c+a'x^2+b'x+c' \in S \Leftrightarrow (a+a')x^2+(b+b')x+(c+c') \in S \Rightarrow c+c'=0 \Leftrightarrow 0=0$

$\forall \alpha \in \mathbb{R}$, $\forall\, ax^2+bx \in S$

$\alpha(ax^2+bx) \in S$?

$ax^2+bx \in S \Rightarrow c=0$

$\alpha(ax^2+bx) \in S \Leftrightarrow (\alpha a)x^2+(\alpha b)x \in S \Rightarrow c=0$

Logo, S é um subespaço vetorial.

(ii) $S = \{ax^2+bx+c \in \mathcal{P}_2 : b=1\} = ax^2+x+c, \forall a,c \in \mathbb{R}$

$0_V \in S$? Sim, para $x=0 \wedge c=0 \Rightarrow 0+0+0=0 \Leftrightarrow 0=0$

$\forall\, ax^2+bx+c , a'x^2+b'x+c' \in S$

$ax^2+bx+c+a'x^2+b'x+c' \in S$?

$ax^2+bx+c \in S \Rightarrow b=1$
$a'x^2+b'x+c' \in S \Rightarrow b'=1$

$ax^2+bx+c + a'x^2+b'x+c' \in S$
$\Leftrightarrow (a+a')x^2+(b+b')x+c+c' \in S \Leftrightarrow b+b'=1 \Leftrightarrow 1+1=1 \Leftrightarrow 2=1$ X

Logo, S não é um subespaço vetorial

(iii)

$S=\{ax^2+bx+c \in \mathcal{P}_2 : bc=0\} = \{ax^2+bx+c \in \mathcal{P}_2 : b=0 \vee c=0\}$
$\hookrightarrow b=0 \vee c=0$

$0_{\mathcal{P}} \in S$? Sim para $x=0 \wedge c=0 \Rightarrow 0+0+0=0 \Leftrightarrow 0=0$

$\forall\ ax^2+bx+c,\ a'x^2+b'x^2+c' \in S$

$ax^2+bx+c+a'x^2+b'x+c' \in S$?

Não, por exemplo para $(b=0 \wedge c=1) \wedge (b=1 \wedge c=0)$:

$ax^2+1 \in S$ e $a'x^2+x \in S$

e $ax^2+1+a'x^2+x = (a+a')x^2+x+1 \notin S$, pois $c\neq 0 \wedge b\neq 0$
$b=1 \wedge c=1$

Logo, S não é um subespaço vetorial.

d)

$\mathcal{V}=\mathbb{R}^{n\times n}$

(i)

$S=\{A \in \mathbb{R}^{n\times n} : A^T=A\}$

$0_{\mathcal{V}}$ é a matriz nula.
$0_{\mathcal{V}} \in S$? Sim, a matriz nula é simétrica.

$\forall X, X' \in S$

$X+X' \in S$?

$X \in S \Leftrightarrow X^T=X$
$X' \in S \Leftrightarrow X'^T=X'$

$X+X' \in S \Leftrightarrow X+X'=(X+X')^T$
$\hookrightarrow$ Verdadeiro, pois $X+X'=X^T+X'^T=(X+X')^T$

$\forall \alpha \in \mathbb{R}, \forall X \in S$

$X \in S \Leftrightarrow X = X^T$

$\alpha X \in S \Leftrightarrow \alpha X = (\alpha X)^T$

Como a transposta de uma constante (α) é igual a α.
$(\alpha X)^T = \alpha X^T$, e como $X = X^T \Rightarrow (\alpha X)^T = \alpha X$

(ii)

$$S = \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{m1} \\ 0 & \alpha_{22} & \cdots & \alpha_{m2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_{mm} \end{bmatrix}, \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mm} \end{bmatrix}, \alpha_{ij} \in \mathbb{R}, i,j \in \{1,2,\dots,m\} \right\}$$

triangular inferior — triangular superior

$0_p \in S$? Sim, a matriz nula é uma matriz triangular ✓

$\forall X, X' \in S$

$X + X' \in S$? Não! Por exemplo: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \in S$ e $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \in S$ e

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \notin S$, porque não é uma matriz triangular

(iii)

$S = \{ A \in \mathbb{R}^{m \times m} : \det(A) \neq 0 \}$

$0_p \in S$? Não pois o determinante de uma matriz nula é zero logo $0_p \notin S$

(iv)

$S = \{ X \in \mathbb{R}^{m \times m} : AX = 0 \}$

$0_p \in S$? Sim, pois um sistema: $A \times 0 = 0 \Leftrightarrow 0 = 0$

$\forall B, B' \in S \quad B + B' \in S$?

$B \in S \Leftrightarrow AB = 0$
$B' \in S \Leftrightarrow AB' = 0$

$B + B' \in S \Leftrightarrow (B + B')X = 0 \Leftrightarrow AB + AB' = 0$ ✓

$\forall \alpha \in \mathbb{R}, \forall B \in S \quad \alpha B \in S?$

$B \in S \Leftrightarrow AB = 0$

$\alpha B \in S \Leftrightarrow A(\alpha B) = 0 \Leftrightarrow \alpha(AB) = 0 \Leftrightarrow \alpha \times 0 = 0 \Leftrightarrow 0 = 0$

Logo, S é s.v.

(v)

$S = \{ X \in \mathbb{R}^{m \times m} : AX = I_m \wedge \det(A) \neq 0 \}$

$0_V \in S$? Não, pois $A \times 0 \neq I_m$, logo $X \notin S$

e)

$S = \{ A \in \mathbb{R}^m : A \cdot X = 0 \}, \forall X \in \mathbb{R}^m$

$0_V \in S$? Sim, pois o vetor nulo é ortogonal a qualquer vetor de $\mathbb{R}^m$ ✓

$\forall B, B' \in S \quad B + B' \in S?$

$B \in S \Leftrightarrow \underline{B \cdot X = 0}$

$B' \in S \Leftrightarrow \underline{B' \cdot X = 0}$

$B + B' \in S \Leftrightarrow (B + B') \cdot X = 0 \Leftrightarrow B \cdot X + B' \cdot X = 0 \Leftrightarrow 0 = 0$ ✓

$\forall \alpha \in \mathbb{R}, \forall B \in S \quad \alpha B \in S?$

$B \in S \Leftrightarrow \underline{B \cdot X = 0}$

$\alpha B \in S \Leftrightarrow (\alpha B) \cdot X = 0 \Leftrightarrow \alpha (B \cdot X) = 0 \Leftrightarrow \alpha \times 0 = 0 \Leftrightarrow 0 = 0$ ✓

Logo, S é um subespaço vetorial

f)

$S = \{ f \in \mathcal{F}(\mathbb{R}) : f(0) = 0 \}$

$0_V = 0 \qquad 0_V \in S$? Sim, por exemplo para $x = 0$. ✓

$\forall g, g' \in S \quad g + g' \in S?$

$g \in S \Leftrightarrow \underline{g(0) = 0}$

$g' \in S \Leftrightarrow \underline{g'(0) = 0}$

$g + g' \in S \Leftrightarrow (g + g')(0) = 0 \Leftrightarrow g(0) + g'(0) = 0 \Leftrightarrow 0 = 0$ ✓

$\forall \alpha \in \mathbb{R}, \forall g \in S \quad \alpha g \in S?$

$g \in S \Leftrightarrow g(0) = 0$

$\alpha g \in S \Leftrightarrow (\alpha g)(0) = 0 \Leftrightarrow \alpha g(0) = 0 \Leftrightarrow 0 = 0$ ✓

Logo, S é um subespaço vetorial

$\boxed{3}$ Caso geral

$$S = \langle X_1, X_2, X_3 \rangle = \{ \alpha_1 X_1, \alpha_2 X_2 + \alpha_3 X_3 : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_3 \}$$

① $0_p = 0X_1 + 0X_2 + 0X_3 \underbrace{\in S}$ ✓

é combinação linear de X_1, X_2, X_3 com $\alpha_1 = \alpha_2 = \alpha_3 = 0$

② Sejam $X, Y \in S$. Vamos ver que $X + Y \in S$

$$X \in S \Leftrightarrow X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

$$Y \in S \Leftrightarrow Y = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3, \quad \beta_1, \beta_2, \beta_3 \in \mathbb{R}$$

$$X + Y = (\alpha_1 + \beta_1) X_1 + (\alpha_2 + \beta_2) X_2 + (\alpha_3 + \beta_3) X_3 = \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3, \text{ com } \theta_i = \alpha_i + \beta_i, \; i \in \{1, 2, 3\}$$

Logo, $X + Y \in S$ ✓

③ $\forall \alpha \in \mathbb{R}, \forall X \in S \qquad \alpha X \in S$?

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

$$\alpha X = (\alpha \alpha_1) X_1 + (\alpha \alpha_2) X_2 + (\alpha \alpha_3) X_3$$

$$= \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_3, \quad \theta_1, \theta_2, \theta_3 \in \mathbb{R}$$

$$\theta_i = \alpha \alpha_i, \; i \in \{1, 2, 3\}$$

Logo, $\alpha X \in S$ ✓

$\boxed{4}$

$$F = \{ P^{-1} A P : A \in E \} = \{ P^{-1} P A : A \in E \}$$

$$= \{ I_n A : A \in E \} = \{ A : A \in E \}$$

Logo, como $A \in E$, que é um subespaço vetorial de $\mathbb{R}^{m \times m}$ por consequência, F é também um subespaço vetorial de $\mathbb{R}^{m \times m}$

$\boxed{5}$

a)

$$(2,-3,-4,3) = \alpha(1,2,1,0) + \beta(4,1,-2,3)$$

$$\begin{cases} \alpha + 4\beta = 2 \\ 2\alpha + \beta = -3 \\ \alpha - 2\beta = -4 \\ 3\beta = 3 \end{cases} \quad \begin{cases} -2+4=2 \\ -4+1=-3 \\ \alpha = -2 \\ \beta = 1 \end{cases} \quad \begin{cases} 2=2 \checkmark \\ -3=-3 \checkmark \\ \alpha=-2 \\ \beta=1 \end{cases}$$

$$(2,-3,-4,3) = -2(1,2,1,0) + (4,1,-2,3)$$

b)

$$\exists \alpha,\beta,\gamma \in \mathbb{R} : (1,1,0) = \alpha(2,1,-2) + \beta(1,0,0) + \gamma(1,1,1)$$

$$\begin{cases} 2\alpha + \beta + \gamma = 1 \\ \alpha + \gamma = 1 \\ -2\alpha + \gamma = 0 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ -2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

$$\begin{cases} \alpha + \gamma = 1 \\ \beta - \gamma = -1 \\ 3\gamma = 2 \end{cases} \quad \begin{cases} \alpha = \frac{1}{3} \\ \beta = -\frac{1}{3} \\ \gamma = \frac{2}{3} \end{cases}$$

Sistema possível e determinado, logo é possível escrever $(1,1,0)$ como comb. linear dos vetores referidos.

$$\text{Logo, } (1,1,0) = \frac{1}{3} \times (2,1,-2) - \frac{1}{3} \times (1,0,0) + \frac{2}{3} \times (1,1,1)$$

c)

$$\exists \alpha,\beta,\gamma \in \mathbb{R} : -t^2 + t + 4 = \alpha(t^2+2t+1) + \beta(t^2+3) + \gamma(t-1)$$

$$\Leftrightarrow \exists \alpha,\beta,\gamma \in \mathbb{R} : -t^2 + t + 4 = t^2(\alpha+\beta) + t(2\alpha+\gamma) + \alpha + 3\beta - \gamma$$

$$\begin{cases} \alpha + \beta = -1 \\ 2\alpha + \gamma = 1 \\ \alpha + 3\beta - \gamma = 4 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & 1 & 3 \\ 0 & 2 & -1 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 0 & 8 \end{array}\right] \longrightarrow$$

Sistema impossível!
Logo é impossível escrever $-t^2+t+4$ como comb. linear dos polinómios dados

d)

$\exists \alpha, \beta, \gamma \in \mathbb{R} : \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \beta \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + \gamma \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

$$\begin{cases} \alpha + \beta + 2\gamma = 1 \\ -\beta + 2\gamma = 1 \\ 2\alpha - \gamma = 0 \\ \alpha + 3\beta + \gamma = 2 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ 1 & 3 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 2 & 5 & 2 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{E.I.}$$

$\boxed{6}$

a) $\{(0,1), (2,1), (2,2)\}$ em $\mathbb{R}^2$

$\langle (0,1), (2,1), (2,2) \rangle$
$= \langle (0,1), (2,1) \rangle = ?$

$\exists \alpha, \beta \in \mathbb{R} : (x,y) = \alpha(0,1) + \beta(2,1)$

$$\begin{cases} 2\beta = x \\ \alpha + \beta = y \end{cases} \Rightarrow \left[\begin{array}{cc|c} 0 & 2 & x \\ 1 & 1 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & y \\ 0 & 2 & x \end{array}\right] \rightarrow \text{Sistema possível e determinado.}$$

Logo, $\langle (0,1), (2,1) \rangle = \mathbb{R}^2$

b) $\{(0,1), (0,2)\}$ em $\mathbb{R}^2$

$\langle (0,1), (0,2) \rangle = ?$

$\exists \alpha, \beta \in \mathbb{R} : (x,y) = \alpha(0,1) + \beta(0,2)$

$$\begin{cases} 0 = x \\ \alpha + 2\beta = y \end{cases} \Rightarrow \left[\begin{array}{cc|c} 0 & 0 & x \\ 1 & 2 & y \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & y \\ 0 & 0 & x \end{array}\right]$$

Sistema possível d
sse $x = 0$

Logo, $\langle (0,1), (0,2) \rangle = \{(x,y) \in \mathbb{R}^2 : x = 0\} = \{(0,y), y \in \mathbb{R}\}$

c) $\{(1,1,1),(1,0,0),(2,2,2)\}$ em $\mathbb{R}^2$

$$\langle (1,1,1),(1,0,0),(2,2,2)\rangle = ?$$

$$\exists \alpha, \beta, \gamma \in \mathbb{R} : (x,y,z) = \alpha(1,1,1) + \beta(1,0,0) + \gamma(2,2,2)$$

$$\begin{cases} \alpha + \beta + 2\gamma = x \\ \alpha + 2\gamma = y \\ \alpha + 2\gamma = z \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 0 & 2 & y \\ 1 & 0 & 2 & z \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & -1 & 0 & y-x \\ 0 & -1 & 0 & z-x \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 0 & -1 & 0 & y-x \\ 0 & 0 & 0 & z-y \end{array}\right]$$, Sistema possível sse $z - y = 0$ $\Leftrightarrow z = y$

$$\langle (1,1,1),(1,0,0),(2,2,2)\rangle = \{(x,y,z) \in \mathbb{R}^3 : z + y = 0\} = \{(x,y,y),\ x,y \in \mathbb{R}\}$$

d)

$$\mathcal{P}_2 = \{ at^2 + bt + c,\ a,b,c \in \mathbb{R} \}$$

$$\langle t^2+1,\ t^2+t,\ t+1 \rangle = ?$$

$$\exists \alpha, \beta, \gamma \in \mathbb{R} : at^2 + bt + c = \alpha(t^2+1) + \beta(t^2+t) + \gamma(t+1)$$
$$\Leftrightarrow \exists \alpha, \beta, \gamma \in \mathbb{R} : at^2 + bt + c = t^2(\alpha+\beta) + t(\beta+\gamma) + \alpha + \gamma$$

$$\begin{cases} \alpha + \beta = a \\ \beta + \gamma = b \\ \alpha + \gamma = c \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & 1 & c-a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & c-a+b \end{array}\right]$$, Sistema Possível e determinado

Logo, $\langle t^2+1,\ t^2+t,\ t+1 \rangle = \mathcal{P}_2$

$\boxed{7}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N(A) = \{ X \in \mathbb{R}^4 : AX = 0 \}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} X_1 + X_3 = 0 \\ X_2 + X_3 + X_4 = 0 \end{cases} \quad \begin{cases} X_1 = -X_3 \\ X_2 = -X_3 - X_4 \end{cases}$$

$$X \in N(A): (X_1, X_2, X_3, X_4) = (-X_3, -X_3 - X_4, X_3, X_4)$$

$$= X_3(-1,-1,1,0) + X_4(0,-1,0,1)$$

$$N(A) = \langle (-1,-1,1,0), (0,-1,0,1) \rangle$$

Logo, um conjunto gerador do espaço nulo de A é: $\{(-1,-1,1,0), (0,-1,0,1)\}$

$\boxed{8}$

a)

$V = \mathbb{R}^2$ e $u = (a,b)$, $a \neq 0 \vee b \neq 0$

$$\langle u \rangle = \langle (a,b) \rangle, \ a \neq 0 \vee b \neq 0 = ?$$

$\exists \alpha \in \mathbb{R}: (x,y) = \alpha(a,b), \ a \neq 0 \vee b \neq 0 \}$

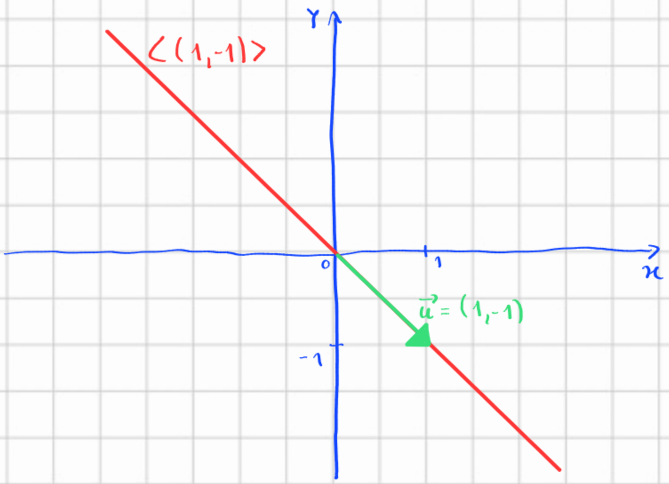
$$\begin{cases} \alpha a = x \\ \alpha b = y \end{cases} \quad \begin{cases} \alpha = \frac{x}{a} \\ y = \alpha b \end{cases}, \ a \neq 0$$

$$r: \quad y = \alpha b \Leftrightarrow y = \frac{x}{a} \times b \Leftrightarrow y = \frac{b}{a} x, \ a \neq 0$$

r tem direção de $u(a,b)$, $m_r = \frac{b}{a}$

Como r é uma função afim, logo passa pela origem.

b)



9

a) $V = \mathbb{R}^3$, u_1 e u_2 de $\mathbb{R}^3$ l.i, $u_1 = (a_1, b_1, c_1)$
$u_2 = (a_2, b_2, c_2)$

- Como u_1 e u_2 são l.i, $u_1 \neq 0 \wedge u_2 \neq 0 \Rightarrow (a_1 \neq 0 \vee b_1 \neq 0 \vee c_1 \neq 0) \wedge (a_2 \neq 0 \vee b_2 \neq 0 \vee c_2 \neq 0)$

$\langle u_1 \rangle = ?$

$\exists \alpha \in \mathbb{R} : (x, y, z) = \alpha(a_1, b_1, c_1)$, $a_1 \neq 0 \vee b_1 \neq 0 \vee c_1 \neq 0$

$\Leftrightarrow \exists \alpha \in \mathbb{R} : \boxed{(x, y, z) = (0, 0, 0) + \alpha(a_1, b_1, c_1)}$, $a_1 \neq 0 \vee b_1 \neq 0 \vee c_1 \neq 0$

Esta expressão define uma reta com direção $(a_1, b_1, c_1) = u_1$, $u_1 \neq 0$ e que passa na origem, quando $\alpha = 0 \Rightarrow (x, y, z) = (0, 0, 0)$

b)

$\langle u_1, u_2 \rangle = ?$

$\exists \alpha, \beta \in \mathbb{R} : (x, y, z) = \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)$, $(a_1, b_1, c_1) \neq 0 \wedge (a_2, b_2, c_2) \neq 0$

$\Leftrightarrow \exists \alpha, \beta \in \mathbb{R} : \boxed{(x, y, z) = (0, 0, 0) + \alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)}$, $(a_1, b_1, c_1) \neq 0 \wedge (a_2, b_2, c_2) \neq 0$

Eq. de um plano que tem como vetores diretores $(a_1, b_1, c_1) = u_1 \neq 0$ e $(a_2, b_2, c_2) = u_2 \neq 0$, logo u_1 e u_2 pertencem ao plano, e que passa pela origem, quando $\alpha = 0 \Rightarrow (x, y, z) = (0, 0, 0)$

c)

(i) $\langle (1, -1, 2) \rangle$ define uma reta que passa na origem, $(0,0,0)$, e com direção de $(1, -1, 2)$

(ii) $\langle (1, 0, 1), (0, 0, 1) \rangle$, como $(1, 0, 1)$ e $(0, 0, 1)$ de $\mathbb{R}^3$ são l.i., define um plano que passa na origem, $(0, 0, 0)$, e tem como vetores diretores $(1, 0, 1)$ e $(0, 0, 1)$

(iii) $\langle (1, -1, 1), (-2, 2, -2) \rangle$, como $(-2, 2, -2) = -1 \times (1, -1, 1)$ logo $(1, -1, 1)$ e $(-2, 2, -2)$ não são l.i. então:

$\langle (1, -1, 1), (-2, 2, -2) \rangle = \langle (1, -1, 1) \rangle$, que define uma reta que passa na origem, $(0, 0, 0)$, e tem direção $(1, -1, 1)$

$\boxed{10}$

a) $\{(1,1,0),(0,2,3),(1,2,3),(1,-1,1)\}$ é l.d., pois por definição, em $\mathbb{R}^3$, qualquer conjunto com mais de três vetores distintos é linearmente dependente.

b)

$\mathcal{B}=\{(1,2,3),(1,1,1),(1,0,1)\}$

$$A=\begin{bmatrix}1&1&1\\2&1&0\\3&1&1\end{bmatrix}$$

O conjunto de vetores $\mathcal{B}$ só é l.i sse $AX=0$ admitir apenas a solução trivial $\Rightarrow \det(A)\neq 0$

$$\begin{vmatrix}1&1&1\\2&1&0\\3&1&1\end{vmatrix}\sim\begin{vmatrix}1&1&1\\2&1&0\\2&0&0\end{vmatrix}=\begin{vmatrix}2&1\\2&0\end{vmatrix}=0-2=-2\neq 0,$$

logo o conjunto de vetores de $\mathcal{B}$ é l.i.

c)

$\mathcal{B}=\{(1,1,1,1),(1,-1,2,3),(1,3,0,-1)\}$

$$A=\begin{bmatrix}1&1&1\\1&-1&3\\1&2&0\\1&3&-1\end{bmatrix}$$

O conjunto de vetores $\mathcal{B}$ só é l.i sse $AX=0$ admitir apenas a solução trivial $\Rightarrow \det(A)\neq 0$

$$\begin{vmatrix}1&1&1\\1&-1&3\\1&2&0\\1&3&-1\end{vmatrix}\sim\begin{bmatrix}1&1&1\\0&-2&2\\0&1&-1\\0&2&-2\end{bmatrix}=\begin{vmatrix}-2&2\\1&-1\\2&-2\end{vmatrix}\sim\begin{vmatrix}-2&2\\0&0\\0&0\end{vmatrix}=0$$

, logo o conjunto de vetores $\mathcal{B}$ é linearmente dependente pois $AX=0$ não admite apenas a solução trivial

d)

$\mathcal{P}_2=\{at^2+bt+c,\ a,b,c\in\mathbb{R}\}$

$\mathcal{B}=\{2t^2+1,\ t-2,\ t+3\}$

$\exists\, \alpha,\beta,\gamma\in\mathbb{R}:\ 0t^2+0t+0=\alpha(2t^2+1)+\beta(t-2)+\gamma(t+3)$

$\Leftrightarrow \exists\, \alpha,\beta,\gamma\in\mathbb{R}:\ 0t^2+0t+0=t^2(2\alpha)+t(\beta+\gamma)+\alpha-2\beta+3\gamma$

$$\begin{cases}2\alpha=0\\\beta+\gamma=0\\\alpha-2\beta+3\gamma=0\end{cases}$$

, o conjunto de vetores de $\mathcal{B}$ é l.i se este sistema apenas admitir a solução trivial $\Rightarrow$ $\begin{vmatrix}2&0&0\\0&1&1\\1&-2&3\end{vmatrix}\neq 0$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = 2\times 5 = 10 \neq 0$$, logo o conjunto de vetores de $\mathcal{B}$ é l.i.

$\boxed{11}$

$\{X_1, \dots, X_m\}$ l.i.

Provar: $\boxed{A \text{ é } n \times n \text{ e invertível} \Longrightarrow \{AX_1, \dots, AX_m\} \text{ é l.i}}$

$$\boxed{\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_m X_m} = 0_p \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_m = 0$$

$\langle X_1, \dots X_m \rangle$

$$A\alpha_1 X_1 + A\alpha_2 X_2 + \dots + A\alpha_m X_m = 0_p$$

$$\Leftrightarrow A(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_m X_m) = 0_p$$

Como A é invertível

$$\Leftrightarrow \cancel{A^{-1}}\left[\cancel{A}(\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_m X_m)\right] = \underbrace{0_p A^{-1}}_{0_p}$$

$$\Leftrightarrow \alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_m X_m = 0_p$$

$\Downarrow \{X_1, X_2, \dots, X_m\}$ são l.i

$$\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$$

$\boxed{12}$

$A \in \mathbb{R}^{2\times 2}$

$\boxed{\text{linhas de A são l.i. sse as colunas de A são l.i}}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

As colunas de A são l.i. sse $AX = 0$ admitir apenas a solução trivial $\Rightarrow \det(A) \neq 0$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

- Seja B a matriz das linhas de A.

$$B = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

As linhas de A são l.i. sse $BX = 0$ admite apenas a solução trivial $\Rightarrow \det(B) \neq 0$

$$\det(B) = \det(A^T) = \det(A)$$

Logo, $\det(B) \neq 0$ sse $\det(A) \neq 0$

Conclui-se assim que as linhas de A são l.i sse as colunas de A forem l.i.

c.q.d.

13

a)

$\mathcal{B} = \{(1,2), (2,4)\}$ em $\mathbb{R}^2$, $V = \mathbb{R}^2$

Como $(2,4) = 2(1,2)$, o conjunto de vetores $\mathcal{B}$ é linearmente dependente, logo não é uma base de $\mathbb{R}^2$

b)

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \text{ em } \mathbb{R}^{2\times 2}, \quad V = \mathbb{R}^{2\times 2}$$

$$\exists \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}: \ \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_3 = 0 \\ \alpha_1 + \alpha_4 = 0 \\ \alpha_2 + \alpha_4 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 = 0 \end{cases}$$

, para o conjunto de vetores ser l.i. este sistema tem de admitir apenas a solução trivial

$$\Rightarrow \begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \neq 0$$

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \sim \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}$$

$= -(0-1) = 1 \neq 0$, logo o conjunto de vetores é l.i.

Como $\dim V = 4$, e o conjunto tem 4 vetores e é l.i logo:

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = \mathbb{R}^{2\times 2}$$

Conclui-se assim que o conjunto de vetores $\mathcal{B}$ é uma base de $\mathbb{R}^{2\times 2}$

c)

$\mathcal{B} = \{(1,0,1), (1,1,0), (0,1,1)\}$ em $\mathbb{R}^3$, $V = \mathbb{R}^3$

Seja $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. O conjunto $\mathcal{B}$ é l.i. sse $AX = 0$ admitir apenas a solução trivial $\Rightarrow \det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \sim \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1-(-1) = 2 \neq 0,$$ logo $\mathcal{B}$ é l.i.

Como $\dim V = 3$ e o conjunto $\mathcal{B}$ tem 3 vetores l.i., conclui-se:

$$\langle (1,0,1), (1,1,0), (0,1,1) \rangle = \mathbb{R}^3$$

d)

$\mathcal{P}_2 = \{at^2 + bt + c, \ a,b,c \in \mathbb{R}\}$

$\mathcal{B} = \{t^2 - 2t + 1, t^2 + t + 1, t^2 + 1\}$ em $\mathcal{P}_2$, $V = \mathcal{P}_2$

$\exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}: 0t^2 + 0t + 0 = \alpha_1(t^2 - 2t + 1) + \alpha_2(t^2 + t + 1) + \alpha_3(t^2 + 1)$

$\Leftrightarrow \exists \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}: 0t^2 + 0t + 0 = t^2(\alpha_1 + \alpha_2 + \alpha_3) + t(-2\alpha_1 + \alpha_2) + \alpha_1 + \alpha_2 + \alpha_3$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases}$$, $\mathcal{B}$ é l.i. sse este sistema admitir apenas a solução trivial $\Longrightarrow \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

$\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$, pois $L_1 = L_2$. Logo, como $\mathcal{B}$ é l.d., conclui-se que $\mathcal{B}$ não é uma base de $\mathcal{P}_2$

$\boxed{14}$

a)

$S = \langle (1,3,0), (-1,1,0) \rangle$, $(1,3,0)$ e $(-1,1,0)$ são l.i

Logo, $\{(1,3,0), (-1,1,0)\}$ é uma base de S

$\begin{bmatrix} 1 & 3 & 0 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \{(1,0,0), (0,1,0)\}$ também é uma base de S

b)

$\mathcal{B} = \{(1,-1,1), (0,2,1), (1,1,2)\}$

$S = \langle (1,-1,1), (0,2,1), (1,1,2) \rangle$

Como $(1,1,2) = 1 \times (1,-1,1) + 1 \times (0,2,1)$, $\mathcal{B}$ não é l.i.

$S = \langle (1,-1,1), (0,2,1) \rangle$, $(1,-1,1)$ e $(0,2,1)$ são l.i.

logo $\{(1,-1,1), (0,2,1)\}$ é uma base de S e a $\dim S = 2$

c)

$\mathcal{B} = \{t^2+1, t^2-t+1\}$ são l.i.

$S = \langle t^2+1, t^2-t+1 \rangle$, logo $\mathcal{B}$ é uma base de S

$$\begin{array}{ccc} t^2 & t & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \{t^2+1, t\} \text{ é outra base de } S$$

$\boxed{15}$

$\mathcal{B} = \{(a^2,0,1), (0,a,2), (1,0,1)\}$

$V = \mathbb{R}^3$, $\dim V = 3$

Sendo $\mathcal{B}$ um conjunto 3 vetores distintos e a $\dim V = 3$, se $\mathcal{B}$ for l.i. $\mathcal{B}$ é uma base de V

$$A = \begin{bmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Para $\mathcal{B}$ ser l.i. o sistema $AX=0$ tem de admitir apenas a solução trivial $\Rightarrow \det(A) \neq 0$

$$\det(A) = \begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{vmatrix} \sim \begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1-a^2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a \\ 1-a^2 & 2 \end{vmatrix} = -a(1-a^2) = -a+a^3$$

$a^3 - a \neq 0$

$\Leftrightarrow a(a^2-1) \neq 0$

$\Leftrightarrow a \neq 0 \wedge a^2 \neq 1$

$\Leftrightarrow a \neq 0 \wedge a \neq -1 \wedge a \neq 1$

R: $a \in \mathbb{R} \setminus \{-1,0,1\}$

$\boxed{16}$

$V = \mathbb{R}^4$, $\dim V = 4$

$\mathcal{B} = \{(1,0,1,0), (0,1,-1,0), (a_1,a_2,a_3,a_4), (b_1,b_2,b_3,b_4)\}$

Como $\mathcal{B}$ é um conjunto de 4 vetores e a $\dim V = 4$, se os vetores de $\mathcal{B}$ forem l.i. $\mathcal{B}$ é uma base de $V = \mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 0 & a_1 & b_1 \\ 0 & 1 & a_2 & b_2 \\ 1 & -1 & a_3 & b_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix}$$

$\mathcal{B}$ é l.i. sse o sistema $AX=0$ admitir apenas a solução trivial $\Rightarrow \det(A) \neq 0$

Por exemplo se $a_1 = 0 \wedge a_2 = 0 \wedge b_1 = 0 \wedge b_2 = 0 \wedge b_3 = 0$
$\wedge a_3 = 1 \wedge b_4 = 1$
$\wedge a_4 = 0$

$a = (a_1,a_2,a_3,a_4) = (0,0,1,0)$
$b = (b_1,b_2,b_3,b_4) = (0,0,0,1)$

$$\Rightarrow \det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1^4 = 1 \neq 0$$

Logo, $\mathcal{B} = \{(1,0,1,0), (0,1,-1,0), (0,0,1,0), (0,0,0,1)\}$ é uma base de $\mathbb{R}^4$

$\boxed{17}$

a)

$S = \{(x,y,z) \in \mathbb{R}^3 : x - y + 3z = 0\}$, $V = \mathbb{R}^3 \Rightarrow 0_V = (0,0,0)$

① - $0_V \in S$? Sim, quando $x = 0 \wedge y = 0 \wedge z = 0 \Rightarrow 0 - 0 + 3 \times 0 = 0 \Leftrightarrow 0 = 0$ ✓
$0_V = (0,0,0)$

②

$\forall (x,y,z), (x',y',z') \in S$

$(x,y,z) + (x',y',z') \in S$?

$(x,y,z) \in S \Leftrightarrow x - y + 3z = 0$
$(x',y',z') \in S \Leftrightarrow x' - y' + 3z' = 0$

$(x+x', y+y', z+z') \in S \Leftrightarrow (x+x') - (y+y') + 3(z+z') = 0$
$\Leftrightarrow x + x' - y - y' + 3z + 3z' = 0$
$\Leftrightarrow (x - y + 3z) + (x' - y' + 3z') = 0$
$\Leftrightarrow 0 + 0 = 0 \Leftrightarrow 0 = 0$ ✓

③ $\forall \alpha \in \mathbb{R}, \forall (x,y,z) \in S$

$\alpha(x,y,z) \in S$?

$(x,y,z) \in S \Leftrightarrow x - y + 3z = 0$

$\alpha(x,y,z) \in S \Leftrightarrow (\alpha x, \alpha y, \alpha z) \in S$

$\Leftrightarrow \alpha x - \alpha y + 3\alpha z = 0$

$\Leftrightarrow \alpha(x - y + 3z) = 0$

$\Leftrightarrow \alpha \times 0 = 0 \Leftrightarrow 0 = 0$ ✓

Logo, S é subespaço de $\mathbb{R}^3$

b)

$S = \{(x,y,z) \in \mathbb{R}^3 : x - y + 3z = 0\}$

$= \{(x,y,z) \in \mathbb{R}^3 : x = y - 3z\}$

$= \{(y-3z, y, z) \quad , \quad y,z \in \mathbb{R}\}$

$= \{y(1,1,0) + z(-3,0,1)\}$

Logo, $\langle (1,1,0), (-3,0,1) \rangle = S$

$\mathcal{B} = \{(1,1,0), (-3,0,1)\}$ é um conjunto gerador de S

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\mathcal{B}$ é l.i. sse $AX=0$ admitir apenas a solução trivial

$$AX=0 \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

→ Sistema possível e determinado

$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$, logo $AX=0$ admite apenas a solução trivial $\Rightarrow \mathcal{B}$ é l.i.

c)

$\dim S = 2$, pois $\mathcal{B}$ é uma base de S e é formado por um conjunto de 2 vetores l.i.

$\boxed{18}$ (Nós assumimos isto na aula como uma proposição Verdadeira, não demonstramos ...) !

a) $\mathcal{B} = \{X_1, X_2, \dots, X_n\}$ é uma base de $V \Rightarrow$ • $\mathcal{B}$ é l.i
• $\mathcal{B}$ gera V

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0, \text{ pois } \mathcal{B} \text{ é l.i}$$

Logo, $\alpha_1(\underbrace{\alpha X_1}) + \alpha_2 X_2 + \dots + \alpha_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

$\alpha \neq 0$, logo $\{\alpha X_1, X_2, \dots, X_n\}$ é l.i.

• Como $\mathcal{B}$ gera V, e V segue as propriedades de um espaço vetorial.

$\forall \alpha \in \mathbb{R}, \forall X_1 \in V \Rightarrow \alpha X_1 \in V$
e $\forall X_1, X_2 \in V \Rightarrow X_1 + X_2 \in V$,
conclui-se assim que se

$$\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n \in V$$

$$\Rightarrow \alpha_1(\underbrace{\alpha X_1}_{\alpha \neq 0}) + \alpha_2 X_2 + \dots + \alpha_n X_n \in V$$

• E como $\mathcal{B}$ é uma base de $V \Rightarrow \dim V = n$

• Como $\mathcal{B}' = \{\alpha X_1, X_2, \dots, X_n\}$ é um conjunto com n vetores l.i. e a $\dim V = n$, logo $\mathcal{B}'$ é uma base de V

b)

(1) $\mathcal{B}$ é l.i., logo $X_1, X_2, \dots, X_n \neq 0$

• $\mathcal{B}' = \{X_1 + X_2 + \dots + X_n, X_2 + X_3 + \dots + X_n, \dots, X_n\}$ é uma base de V sse, $\mathcal{B}'$ é l.i. e gera V

$$A = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ X_2 & X_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ X_n & X_n & \cdots & \cdots & X_n \end{bmatrix}$$

• $\mathcal{B}$ é l.i. sse $AX = 0$ admite apenas a solução trivial $\Rightarrow \det(A) \neq 0$

$$\det(A) = \begin{vmatrix} X_1 & 0 & 0 & \cdots & 0 \\ X_2 & X_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ X_n & X_n & \cdots & \cdots & X_n \end{vmatrix} \Rightarrow \det(A) = X_1 \times X_2 \times X_3 \times \dots \times X_n \neq 0,$$

Matriz triângular superior

devido a (1)

• E como $\mathcal{B}$ é uma base de $V \Rightarrow \dim V = n$

• Como $\mathcal{B}'$ é um conjunto de n vetores l.i e $\dim V = n$, logo $\mathcal{B}'$ é uma base de V

$\boxed{19}$

a)

$N(A) = \{ X \in \mathbb{R}^3 : AX = 0 \}$

dim $N(A)$ = nul(A) = n - car(A)
= nº de inc. livres

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in N(A) \Leftrightarrow AX = 0$

$$\Rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{matrix} & x_2 & & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

nul(A) = n - car(A)
= 4 - 2
= 2
= dim $N(A)$

$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ -x_3 + x_4 = 0 \end{cases}$ $\begin{cases} x_1 = -x_2 - 2x_4 \\ x_3 = x_4 \end{cases}$

$N(A) = \{ (x_1, x_2, x_3, x_4),\ x_1 = -x_2 - 2x_4 \wedge x_3 = x_4 \}$

$= \{ (-x_2 - 2x_4, x_2, x_4, x_4),\ x_2, x_4 \in \mathbb{R} \}$

$= \{ x_2(-1, 1, 0, 0) + x_4(-2, 0, 1, 1),\ x_2, x_4 \in \mathbb{R} \}$

$N(A) = \langle (-1, 1, 0, 0), (-2, 0, 1, 1) \rangle$, dim $N(A) = 2$

Logo, como $\mathcal{B} = \{ (-1,1,0,0), (-2,0,1,1) \}$ é um conjunto de 2 vetores l.i. que pertencem a $N(A)$ e dim $N(A) = 2$, logo $\mathcal{B}$ é uma base de $N(A)$

b)

$S = \{ \overbrace{AX}^{\mathcal{C}(A)} : X \in \mathbb{R}^4 \}$

$\mathcal{C}(A)$ = espaço das colunas de A

$B \in S \Leftrightarrow B = AX$ é P.D., $X \in \mathbb{R}^4$

$\Leftrightarrow B \in \mathcal{C}(A)$

$\Leftrightarrow B = x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4$

$B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$[A|B] = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & a \\ 0 & 0 & -1 & 1 & b \\ 1 & 1 & 0 & 2 & c \end{array}\right] \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & a \\ 0 & 0 & -1 & 1 & b \\ 0 & 0 & 0 & 0 & c-a-2b \end{array}\right]$$

$[A|B]$ é possível e indeterminado sse $c - a - 2b = 0$

$S = \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : c - a - 2b = 0 \}$

$= \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : c = a + 2b \}$

$= \{ \begin{bmatrix} a \\ b \\ a+2b \end{bmatrix}, a, b \in \mathbb{R} \} = \{ a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, a, b \in \mathbb{R} \}$

Uma base de S e dim S = 2 = car(A)

$S = \langle (1,0,1), (0,1,2) \rangle$

c)

$S = \langle (1,-1,-1),(4,-3,-2) \rangle$?

$\boxed{\begin{matrix}(1,-1,-1) \in S \\ (4,-3,-2) \in S\end{matrix}}$, $a = 1 \wedge b = -1 \wedge c = -1 \Rightarrow -1 = 1 + 2 \times (-1) \Leftrightarrow -1 = -1$ ✓

$a = 4 \wedge b = -3 \wedge c = -2 \Rightarrow -2 = 4 + 2 \times (-3) \Leftrightarrow -2 = -2$ ✓

• $\mathcal{B} = \{(1,-1,-1),(4,-3,-2)\}$ são l.i.?

$\alpha_1(1,-1,-1) + \alpha_2(4,-3,-2) = (0,0,0)$

$\Downarrow$

$\boxed{\alpha_1 = \alpha_2 = 0}$

$\begin{cases} \alpha_1 + 4\alpha_2 = 0 \\ -\alpha_1 - 3\alpha_2 = 0 \\ -\alpha_1 - 2\alpha_2 = 0 \end{cases} \quad \begin{cases} \text{———} \\ \alpha_1 = -3\alpha_2 \\ \alpha_2 = 0 \end{cases} \quad \begin{cases} \alpha_2 = 0 \\ \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases}$ ✓

• Logo, como $\mathcal{B}$ é um conjunto de 2 vetores l.i. que pertencem a S e $\dim S = 2$, logo $\mathcal{B}$ é uma base de S

$\boxed{21}$

a)

(i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

$N(A) = \{X \in \mathbb{R}^3 : AX = 0\}$

$AX = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -3 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} X_1 = 0 \\ X_2 = 0 \\ X_3 = 0 \end{cases}$

$N(A) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0\}$

$= \{(0,0,0)\} = 0_V$

$\dim N(A) = \dim 0_V = 0$, logo $N(A)$ tem como base $\emptyset$

(ii) $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Logo, $\mathcal{L}(A) = \langle (1,0,0),(0,1,0),(0,0,1) \rangle$, sendo $\{(1,0,0),(0,1,0),(0,0,1)\}$ uma base do espaço das linhas de A

$\mathcal{C}(A) = \langle (1,2,-3,1), (0,1,2,2), (1,-1,3,3) \rangle$, sendo $\{(1,2,-3,1), (0,1,2,2), (1,-1,3,3)\}$ uma base do espaço das colunas de A

(iii)

$car(A) = \dim \mathcal{L}(A) = \dim \mathcal{C}(A) = 3$

$nul(A) = \dim \mathcal{N}(A) = 0$

$n = 3+0 = 3$ ✓

(iv)

Como $car(A) = \dim \mathcal{L}(A) = 3$, logo o máximo de colunas l.i. da matriz A é 3, logo as linhas de A, que são 4, são linearmente dependentes

b)

(i) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 4 & -1 & -2 \end{bmatrix}$

$\mathcal{N}(A) = \{X \in \mathbb{R}^4 : AX = 0\}$

$AX = 0 \Leftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 3 & 4 & -1 & -2 & 0 \end{array}\right] \sim \begin{array}{ccccc} & & x_3 & x_4 & \\ & & \downarrow & \downarrow & \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 4 & -7 & -5 & 0 \end{array}\right]$

$\begin{cases} x_1 + 2x_3 + x_4 = 0 \\ 4x_2 - 7x_3 - 5x_4 = 0 \end{cases} \quad \begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = \frac{7}{4}x_3 + \frac{5}{4}x_4 \end{cases}$

$\mathcal{N}(A) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = -2x_3 - x_4 \wedge x_2 = \frac{7}{4}x_3 + \frac{5}{4}x_4\}$

$= \{(-2x_3 - x_4, \frac{7}{4}x_3 + \frac{5}{4}x_4, x_3, x_4), x_3, x_4 \in \mathbb{R}\}$

$= \{x_3(-2, \frac{7}{4}, 1, 0) + x_4(-1, \frac{5}{4}, 0, 1), x_3, x_4 \in \mathbb{R}\}$

$= \langle (-2, \frac{7}{4}, 1, 0), (-1, \frac{5}{4}, 0, 1) \rangle$, $\dim \mathcal{N}(A) = 2$

Sendo, $\{(-2, \frac{7}{4}, 1, 0), (-1, \frac{5}{4}, 0, 1)\}$ uma base de $\mathcal{N}(A)$

(ii)

$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 4 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -7 & -5 \end{bmatrix}$

$\mathcal{L}(A) = \langle (1,0,2,1), (0,4,-7,-5) \rangle$, sendo $\{(1,0,2,1), (0,4,-7,-5)\}$ uma base de $\mathcal{L}(A)$

$\mathcal{C}(A) = \langle (1,3), (0,4) \rangle$, sendo $\{(1,3), (0,4)\}$ uma base de $\mathcal{C}(A)$

(iii)

$car(A) = \dim \mathcal{L}(A) = \dim \mathcal{C}(A) = 2$

$nul(A) = \dim \mathcal{N}(A) = 2$

$car(A) + nul(A) = n$
$\Leftrightarrow 2 + 2 = 4 \Leftrightarrow 4 = 4$ ✓

(iv)

Como $car(A) = \dim \mathcal{L}(A) = 2$ e a matriz A tem 2 linhas, logo as linhas de A são l.i.

Muito grande ‼

c)

(i)

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & 0 & -1 & 2 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix} \qquad AX = 0 \Leftrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 3 & 1 & 0 & -1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\mathcal{N}(A) = \{X \in \mathbb{R}^5 : AX = 0\}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 5 & 9 & 7 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 5 & 9 & 7 & 1 & 0 \\ 0 & 0 & 13 & 9 & -3 & 0 \end{array}\right]$$

$$(\ldots) \sim \begin{array}{c} \begin{array}{cccccc} & & & x_4 & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -5/13 & 6/13 & 0 \\ 0 & 1 & 0 & 2/13 & 8/13 & 0 \\ 0 & 0 & 1 & 9/13 & -3/13 & 0 \end{array}\right] \end{array} \quad (\ldots) \quad \begin{cases} x_1 = \frac{5}{13}x_4 - \frac{6}{13}x_5 \\ x_2 = -\frac{2}{13}x_4 - \frac{8}{13}x_5 \\ x_3 = -\frac{9}{13}x_4 + \frac{3}{13}x_5 \end{cases}$$

$$\mathcal{N}(A) = \left\{\left(\frac{5}{13}x_4 - \frac{6}{13}x_5, -\frac{2}{13}x_4 - \frac{8}{13}x_5, -\frac{9}{13}x_4 + \frac{3}{13}x_5, x_4, x_5\right), x_4, x_5 \in \mathbb{R}\right\}$$

$$= \left\{x_4 \times \left(\frac{5}{13}, -\frac{2}{13}, -\frac{9}{13}, 1, 0\right) + x_5\left(-\frac{6}{13}, -\frac{8}{13}, \frac{3}{13}, 0, 1\right), x_4, x_5 \in \mathbb{R}\right\}$$

$$= \left\langle \left(\frac{5}{13}, -\frac{2}{13}, -\frac{9}{13}, 1, 0\right), \left(-\frac{6}{13}, -\frac{8}{13}, \frac{3}{13}, 0, 1\right) \right\rangle$$

$$= \langle (5, -2, -9, 13, 0), (-6, -8, 3, 0, 13) \rangle$$

sendo $\{(5,-2,-9,13,0), (-6,-8,3,0,13)\}$ uma base de $\mathcal{N}(A)$

(ii)

$$A \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -5/13 & 6/13 & 0 \\ 0 & 1 & 0 & 2/13 & 8/13 & 0 \\ 0 & 0 & 1 & 9/13 & -3/13 & 0 \end{array}\right]$$

$\mathcal{L}(A) = \langle (13,0,0,-5,6), (0,13,0,2,8), (0,0,13,9,-3) \rangle$
sendo $\{(13,0,0,-5,6), (0,13,0,2,8), (0,0,13,9,-3)\}$ uma base de $\mathcal{L}(A)$

$\mathcal{C}(A) = \langle (1,3,0), (2,1,3), (3,0,1) \rangle$,
sendo $\{(1,3,0), (2,1,3), (3,0,1)\}$ uma base de $\mathcal{C}(A)$

(iii)

$car(A) = \dim \mathcal{L}(A) = \dim \mathcal{C}(A) = 3$
$nul(A) = \dim \mathcal{N}(A) = 2$

$\Rightarrow car(A) + nul(A) = n$
$\Leftrightarrow 3 + 2 = 5 \Leftrightarrow 5 = 5$ ✓

(iv)

Como $car(A) = \dim \mathcal{L}(A) = 3$, conclui-se que 3 das linhas de A são linearmente independentes.
Como A tem 3 linhas, logo as linhas de A são l.i.

d)

(i)

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathcal{N}(A) = \{X \in \mathbb{R}^3 : AX = 0\}$$

$$AX = 0 \Leftrightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow \mathcal{N}(A) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 0 \wedge x_2 = 0 \wedge x_3 = 0\}$$
$$= \{(0,0,0)\} = 0_{\mathbb{R}}$$

$$\dim \mathcal{N}(A) = \dim 0_{\mathbb{R}} = 0$$

Logo, $\mathcal{N}(A)$ tem como base o $\emptyset$

(ii)

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathcal{L}(A) = \langle (1,0,0), (0,1,0), (0,0,1) \rangle$, sendo $\{(1,0,0), (0,1,0), (0,0,1)\}$ uma base de $\mathcal{L}(A)$

$\mathcal{C}(A) = \langle (1,-1,0), (2,2,1), (-3,3,1) \rangle$, sendo $\{(1,-1,0), (2,2,1), (-3,3,1)\}$ uma base de $\mathcal{C}(A)$

(iii)

$car(A) = \dim \mathcal{L}(A) = \dim \mathcal{C}(A) = 3$
$nul(A) = \dim \mathcal{N}(A) = 0$

$car(A) + nul(A) = n$
$\Leftrightarrow 3 + 0 = 3 \Leftrightarrow 3 = 3$ ✓

(iv)

Como $car(A) = \dim \mathcal{L}(A) = 3$, conclui-se que 3 das linhas de A são l.i.
Logo, como a matriz A tem 3 linhas, as linhas de A são l.i.

$\boxed{22}$

$$\mathcal{C}(A) = \mathcal{L}(A^T)$$

$$\mathcal{C}(AB) = \mathcal{L}((AB)^T) = \mathcal{L}(A^T B^T) = \mathcal{L}(A^T) \cap \mathcal{L}(B^T)$$
$$= \mathcal{C}(A) \cap \mathcal{L}(B^T)$$

$$\mathcal{C}(AB) = \mathcal{C}(A) \cap \mathcal{L}(B^T)$$

Logo $\mathcal{C}(AB) \subset \mathcal{C}(A)$

$\boxed{23}$

$$\mathcal{B}_1 = ((1,2,1), (0,2,0), (0,0,-1))$$
$$\mathcal{B}_2 = ((1,0,-1), (1,1,1), (2,3,-1))$$

a)

(i)

$$(2,3,5) = \alpha_1 (1,2,1) + \alpha_2 (0,2,0) + \alpha_3 (0,0,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 3 \\ 1 & 0 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\Rightarrow [(2,3,5)]_{\mathcal{B}_1} = \begin{bmatrix} 2 \\ -1/2 \\ -3 \end{bmatrix}$$

$$(2,3,5) = \alpha_1' (1,0,-1) + \alpha_2' (1,1,1) + \alpha_3' (2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ -1 & 1 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6/5 \\ 0 & 1 & 0 & 18/5 \\ 0 & 0 & 1 & -1/5 \end{array}\right] \Rightarrow [(2,3,5)]_{\mathcal{B}_2} = \begin{bmatrix} -6/5 \\ 18/5 \\ -1/5 \end{bmatrix}$$

(ii)

$$(-1,2,0) = \alpha_1 (1,2,1) + \alpha_2 (0,2,0) + \alpha_3 (0,0,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 2 & 2 & 0 & 2 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$[(-1,2,0)]_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$(-1,2,0) = \alpha_1(1,0,-1) + \alpha_2(1,1,1) + \alpha_3(2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ -1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{cases} \alpha_1 = -2 \\ \alpha_2 = -1 \\ \alpha_3 = 1 \end{cases}$$

$$[(-1,2,0)]_{\beta_2} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

(iii)

$$(1,1,1) = \alpha_1(1,2,1) + \alpha_2(0,2,0) + \alpha_3(0,0,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} \alpha_1 = 1 \\ \alpha_2 = -1/2 \\ \alpha_3 = 0 \end{cases} \Rightarrow [(1,1,1)]_{\beta_1} = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$(1,1,1) = \alpha_1(1,0,-1) + \alpha_2(1,1,1) + \alpha_3(2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 1 \\ \alpha_3 = 0 \end{cases} \Rightarrow [(1,1,1)]_{\beta_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

b)

$$M(\beta_1, \beta_2) = \left[\, [(1,2,1)]_{\beta_2},\ [(0,2,0)]_{\beta_2},\ [(0,0,-1)]_{\beta_3} \right]$$

$$(1,2,1) = \alpha_1(1,0,-1) + \alpha_2(1,1,1) + \alpha_3(2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ -1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 2/5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/5 \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & 2/5 \end{array}\right] \quad \begin{cases} \alpha_1 = -3/5 \\ \alpha_2 = 4/5 \\ \alpha_3 = 2/5 \end{cases} \Rightarrow [(1,2,1)]_{\beta_2} = \begin{bmatrix} -3/5 \\ 4/5 \\ 2/5 \end{bmatrix}$$

$$(0,2,0) = \alpha_1(1,0,-1) + \alpha_2(1,1,1) + \alpha_3(2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ -1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 4/5 \end{array}\right] \quad \begin{cases} \alpha_1 = -6/5 \\ \alpha_2 = -2/5 \\ \alpha_3 = 4/5 \end{cases} \quad [(0,2,0)]_{\beta_2} = \begin{bmatrix} -6/5 \\ -2/5 \\ 4/5 \end{bmatrix}$$

$$(0,0,-1) = \alpha_1(1,0,-1) + \alpha_2(1,1,1) + \alpha_3(2,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/5 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & 1/5 \end{array}\right] \quad \begin{cases} \alpha_1 = 1/5 \\ \alpha_2 = -3/5 \\ \alpha_3 = 1/5 \end{cases} \Rightarrow [(0,0,-1)]_{\beta_2} = \begin{bmatrix} 1/5 \\ -3/5 \\ 1/5 \end{bmatrix}$$

$$M(\beta_1, \beta_2) = \begin{bmatrix} -3/5 & -6/5 & 1/5 \\ 4/5 & -2/5 & -3/5 \\ 2/5 & 4/5 & 1/5 \end{bmatrix} = \frac{1}{5} \times \begin{bmatrix} -3 & -6 & 1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$[(2,3,5)]_{\beta_2} = M(\beta_1,\beta_2) \times [(2,3,5)]_{\beta_1}$$

$$\Leftrightarrow [(2,3,5)]_{\beta_2} = \frac{1}{5} \times \begin{bmatrix} -3 & -6 & 1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1/2 \\ -3 \end{bmatrix}$$

$$\Leftrightarrow [(2,3,5)]_{\beta_2} = \frac{1}{5} \times \begin{bmatrix} -6 \\ 18 \\ -1 \end{bmatrix} \Leftrightarrow [(2,3,5)]_{\beta_2} = \begin{bmatrix} -6/5 \\ 18/5 \\ -1/5 \end{bmatrix}$$

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a) $[(1,5)]_S = [\alpha_1(1,2) + \alpha_2(0,1)]_S$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right] \quad \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 3 \end{cases}$$

$$[(1,5)]_S = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (1,3)$$

$(1,5) = \alpha'_1(1,1) + \alpha'_2(2,3)$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 3 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 4 \end{array}\right] \quad \begin{cases} \alpha'_1 = -7 \\ \alpha'_2 = 4 \end{cases}$$

$$[(1,5)]_{\gamma} = \begin{bmatrix} -7 \\ 4 \end{bmatrix} = (-7,4)$$

b)

$$[z]_{\mathcal{T}} = (1,-3)$$

$$z = (a,b)$$

$$[(a,b)]_{\mathcal{T}} = [1(1,1) + (-3)(2,3)]_{\mathcal{T}}$$

$$= [(-5,-8)]_{\mathcal{T}}$$

$$z = (-5,-8)$$

c)

$$M(\mathcal{T},S) = \left[\ [(1,1)]_S,\ [(2,3)]_S \right]$$

$$(1,1) = \alpha_1(1,2) + \alpha_2(0,1)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{cases} \alpha_1 = 1 \\ \alpha_2 = -1 \end{cases} \Rightarrow [(1,1)]_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(2,3) = \alpha'_1(1,2) + \alpha'_2(0,1)$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

$$\begin{cases} \alpha'_1 = 2 \\ \alpha'_2 = -1 \end{cases} \Rightarrow [(2,3)]_S = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$M = M(\mathcal{T},S) = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

d)

$$[(1,5)]_S = M(\mathcal{T},S) \times [(1,5)]_{\mathcal{T}}$$

já sabiamos

$$= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Inversa, 2x2:

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

Inverte-se os outros

Troca-se DP

e)

$$N = M(S,\mathcal{T}) = M^{-1}(\mathcal{T},S) = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

f)

$$[(1,5)]_{\mathcal{T}} = M(S,\mathcal{T}) \times [(1,5)]_S$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix} \checkmark$$

25

$S = ((-1,1,0),(1,0,1),(0,0,1))$, $J = (Y_1, Y_2, Y_3)$

$$M(J,S) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = [(1,2,-1),(1,1,-1),(2,1,1)]$$

$[Y_1]_S = (1,2,-1)$

$Y_1 = 1(-1,1,0) + 2(1,0,1) - 1(0,0,1) = (1,1,1)$

$[Y_2]_S = (1,1,-1)$

$Y_2 = 1(-1,1,0) + 1(1,0,1) - 1(0,0,1) = (0,1,0)$

$[Y_3]_S = (2,1,1)$

$Y_3 = 2(-1,1,0) + 1(0,0,1) + 1(0,0,1) = (-2,2,2)$

$Y = ((1,1,1),(0,1,0),(-2,2,2))$

26

a) $\{(1,2,1),(0,-1,2),(0,2,1)\}$

São ortogonais sse $(1,2,1)\cdot(0,-1,2) = 0 \Leftrightarrow -2+2=0$
$(1,2,1)\cdot(0,2,1) = 0 \Leftrightarrow 5=0$ X , logo não é ortogonal
$(0,-1,2)\cdot(0,2,1)=0 \Leftrightarrow -2+2=0$

b) $\{(1,2,-1,1),(0,-1,-2,0),(1,0,0,-1)\}$

São ortogonais sse $(1,2,-1,1)\cdot(0,-1,-2,0) = 0 \Leftrightarrow -2+2=0 \Leftrightarrow 0=0$
$(1,2,-1,1)\cdot(1,0,0,-1) = 0 \Leftrightarrow 1-1=0 \Leftrightarrow 0=0$, logo é ortogonal
$(0,-1,-2,0)\cdot(1,0,0,-1)=0 \Leftrightarrow 0=0$

27

$\left\{\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right),\left(a,\frac{\sqrt{2}}{2},-b\right)\right\}$ é ortonormado sse

$\left\|\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\| = \left\|\left(a,\frac{\sqrt{2}}{2},-b\right)\right\| = 1 \ \wedge \ \left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\cdot\left(a,\frac{\sqrt{2}}{2},-b\right) = 0$

$\|(1,0,1)\| = \sqrt{2}$

$\left\|\left(a,\frac{\sqrt{2}}{2},-b\right)\right\| = 1 \Leftrightarrow \sqrt{a^2+\frac{1}{2}+b^2} = 1$

$\left\|\frac{\sqrt{2}}{2}\times(1,0,1)\right\| = \frac{\sqrt{2}}{2}\times\sqrt{2} = 1$ ✓

$\Leftrightarrow \boxed{a^2+b^2 = \frac{1}{2}}$

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(a, \frac{\sqrt{2}}{2}, -b\right) = 0$$

$$\Leftrightarrow a\frac{\sqrt{2}}{2} - b\frac{\sqrt{2}}{2} = 0$$

$$\Leftrightarrow a = b$$

$$\left\|\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)\right\| = \left\|\left(a, \frac{\sqrt{2}}{2}, -b\right)\right\| = 1 \ \wedge \ \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(a, \frac{\sqrt{2}}{2}, -b\right) = 0$$

$$\Rightarrow \begin{cases} a^2 + b^2 = 1/2 \\ a = b \end{cases} \quad \begin{cases} a^2 = \frac{1}{4} \\ b = a \end{cases}$$

$$\Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = -\frac{1}{2} \end{cases} \vee \begin{cases} a = \frac{1}{2} \\ b = \frac{1}{2} \end{cases}$$

O conjunto é ortonormado sse $\left(a = -\frac{1}{2} \wedge b = -\frac{1}{2}\right) \vee \left(a = \frac{1}{2} \wedge b = \frac{1}{2}\right)$

$\boxed{28}$

a)

$$B = \left(\left(\frac{4}{5}, 0, \frac{3}{5}\right), (0,1,0), \left(-\frac{3}{5}, 0, \frac{4}{5}\right)\right)$$

O conjunto $\mathcal{B} = \{X_1, X_2, X_3\}$ é ortogonal sse $(4/5, 0, 3/5) \cdot (0,1,0) = 0 \Leftrightarrow 0 = 0$

$(4/5, 0, 3/5) \cdot (-3/5, 0, 4/5) = 0 \Leftrightarrow -\frac{12}{5} + \frac{12}{5} = 0$, logo $\mathcal{B}$ é

$(0,1,0) \cdot (-3/5, 0, 4/5) = 0 \Leftrightarrow 0 = 0$ ortogonal

O conjunto $\mathcal{B}$, ortogonal, é ortonormado sse:

$\|X_1\| = 1$

$\Leftrightarrow \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$

$\Leftrightarrow 1 = 1$ ✓

$\|X_2\| = 1$

$\Leftrightarrow \sqrt{1} = 1$

$\Leftrightarrow 1 = 1$ ✓

$\|X_3\| = 1$

$\Leftrightarrow \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$

$\Leftrightarrow 1 = 1$ ✓

, logo $\mathcal{B}$ é ortonormado

Como B é uma base formada por 3 vetores o.n. (logo l.i.) e $V = \mathbb{R}^3 \Rightarrow \dim V = 3$, $\# B = \dim \mathbb{R}^3 = 3$, logo B é uma base ortonormada de $\mathbb{R}^3$

b)

$$[X]_B = \begin{bmatrix} X \cdot X_1 \\ X \cdot X_2 \\ X \cdot X_3 \end{bmatrix}$$

$$B = (X_1, X_2, X_3)$$

$$[(1,1,1)]_B = \begin{bmatrix} (1,1,1)\cdot(\frac{4}{3},0,3/5) \\ (1,1,1)\cdot(0,1,0) \\ (1,1,1)\cdot(-\frac{3}{5},0,\frac{4}{5}) \end{bmatrix}$$

$$[(1,1,1)]_B = \begin{bmatrix} 7/5 \\ 1 \\ 1/5 \end{bmatrix}$$

$$[X]_B = \begin{bmatrix} X \cdot Y_1 \\ X \cdot X_2 \\ X \cdot X_3 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 1 \\ 1/5 \end{bmatrix}$$

$$X \cdot X_1 = (1,1,1)\cdot(\frac{4}{3},0,\frac{3}{5}) = 7/5$$

$$X \cdot X_2 = (1,1,1)\cdot(0,1,0) = 1$$

$$X \cdot X_3 = (1,1,1)\cdot(-\frac{3}{5},0,\frac{4}{5}) = \frac{1}{5}$$

c)

$$M(\tilde{B}, B) = \left[[(0,0,1)]_B \; [(0,1,1)]_B \; [(1,1,1)]_B \right]$$

$$[(1,1,1)]_B = \begin{bmatrix} 7/5 \\ 1 \\ 1/5 \end{bmatrix}$$

$$[(0,1,1)]_B = \begin{bmatrix} 3/5 \\ 1 \\ 4/5 \end{bmatrix}$$

$$[(0,0,1)]_B = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

Logo, $M(\tilde{B}, B) = \begin{bmatrix} 3/5 & 3/5 & 7/5 \\ 0 & 1 & 1 \\ 4/5 & 4/5 & 1/5 \end{bmatrix} = \frac{1}{5} \times \begin{bmatrix} 3 & 3 & 7 \\ 0 & 5 & 5 \\ 4 & 4 & 1 \end{bmatrix}$

d)

$$[Y]_{\tilde{B}} = (1,2,3)$$

$$[Y]_B = M(\tilde{B},B)\cdot(1,2,3) = \frac{1}{5} \times \begin{bmatrix} 3 & 3 & 7 \\ 0 & 5 & 5 \\ 4 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$[Y]_B = \frac{1}{5} \times \begin{bmatrix} 30 \\ 25 \\ 15 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} = (6,5,3)$$

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$$X \cdot Y_1 = 0$$
$$X \cdot Y_2 = 0$$
$$\vdots$$
$$X \cdot Y_m = 0$$

$S = \langle \underbrace{Y_1, Y_2, \dots, Y_m} \rangle = \alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_m Y_m$

X é ortogonal a qualquer vetor de S sse: $X \cdot S = 0$

$$X \cdot (\alpha_1 Y_1 + \alpha_2 Y_2 + \dots + \alpha_m Y_m) = \alpha_1 \underbrace{(X \cdot Y_1)}_{0} + \alpha_2 \underbrace{(X \cdot Y_2)}_{0} + \dots + \alpha_m \underbrace{(X \cdot Y_m)}_{0}$$

$= \alpha_1 \times \boxed{0} + \alpha_2 \times \boxed{0} + \dots + \alpha_m \times \boxed{0} = 0$, logo X é ortogonal a qualquer vetor de S

$\boxed{30}$

a)

$$\mathcal{P} = \langle X_1, X_2 \rangle = \langle (1,1,0), (0,0,1) \rangle$$

$X_1 \cdot X_2 = (1,1,0) \cdot (0,0,1) = 0$, logo X_1 e X_2 são ortogonais

$X_1 \cdot X_2 = 0 \Rightarrow X_1$ e X_2 são l.i.

$$\Downarrow$$

$\{X_1 \text{ e } X_2\}$ é uma base $\Rightarrow \dim \mathcal{P} = 2$

Logo, quaisqueres 2 vetores ortonormados pertencentes a $\mathcal{P}$ formam uma base ortonormada de $\mathcal{P}$.

Por exemplo: $Y_1 = \frac{\sqrt{2}}{2} \times (1,1,0) + 0 \times (0,0,1)$
$= \frac{\sqrt{2}}{2} \times (1,1,0)$

$Y_2 = 0 \times (1,1,0) + 1(0,0,1)$
$= (0,0,1)$

$Y_1 \cdot Y_2 = \frac{\sqrt{2}}{2} \times (1,1,0) \cdot (0,0,1) = \frac{\sqrt{2}}{2} \times 0 = 0$, logo $\{Y_1, Y_2\}$ é ortogonal

$|(1,1,0)| = \sqrt{2}$ $\qquad |(0,0,1)| = \sqrt{1} = 1$ ✓

$|Y_1| = \left| \frac{\sqrt{2}}{2} \times (1,1,0) \right| = \frac{\sqrt{2}}{2} \times \sqrt{2} = 1$ ✓

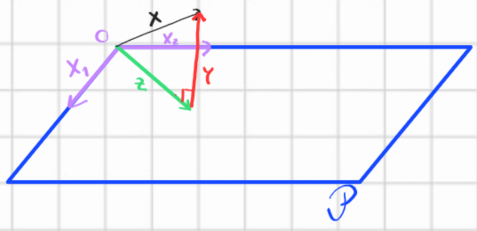
Logo, como $Y_1 \cdot Y_2 = 0 \wedge |Y_1| = 0 \wedge |Y_2| = 0$, $\{Y_1, Y_2\}$ é ortonormado.

Sendo $\{Y_1, Y_2\}$ ortonormado e um conjunto de 2 vetores ($\#\{Y_1, Y_2\} = \dim \mathcal{P} = 2$) $\{Y_1, Y_2\}$ é uma base ortonormada de $\mathcal{P}$.

b)

$\left\{ \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), (0,0,1) \right\}$ é um base ortonormada de $\mathcal{P}$

$$\mathcal{P} = \left\langle \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), (0,0,1) \right\rangle = \left\{ \alpha_1 \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) + \alpha_2 (0,0,1), \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$



$X_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$, $\|X_1\| = 1$
$X_2 = (0,0,1)$, $\|X_2\| = 1$

$Z = \alpha_1 X_1 + \alpha_2 X_2$

$X = Y + Z = Y + \alpha_1 X_1 + \alpha_2 X_2$

$$X \cdot X_1 = \underbrace{Y X_1}_{0} + \alpha_1 \underbrace{(X_1 \cdot X_1)}_{1} + \alpha_2 \underbrace{(X_2 \cdot X_1)}_{0} = \alpha_1$$

$$X \cdot X_2 = \underbrace{Y X_2}_{0} + \alpha_1 \underbrace{(X_1 \cdot X_2)}_{0} + \alpha_2 \underbrace{(X_2 \cdot X_2)}_{1} = \alpha_2$$

$$z = proj_W X = \alpha_1 X_1 + \alpha_2 X_2 = (X \cdot X_1) X_1 + (X \cdot X_2) X_2$$

Neste caso, $X = (2,-2,1)$:

$$proj_W X = \left((2,-2,1)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right)\right)\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right) + \left((2,-2,1)\cdot(0,0,1)\right)(0,0,1)$$

$$= 0\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right) + 1\,(0,0,1)$$

$$= (0,0,1)$$

c)

$$X = Y + Z \Leftrightarrow Y = X - Z$$

$X = (2,1,1)$

$$Z = proj_{\mathcal{B}} X = \left((2,1,1)\cdot(\sqrt{2}/2, \sqrt{2}/2, 0)\right)(\sqrt{2}/2, \sqrt{2}/2, 0) + \left((2,1,1)\cdot(0,0,1)\right)(0,0,1)$$

$$= \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right) + (0,0,1)$$

$$= \left(\frac{3}{2},\frac{3}{2},1\right)$$

$$Y = (2,1,1) - \left(\frac{3}{2},\frac{3}{2},1\right) = \left(\frac{1}{2},-\frac{1}{2},0\right)$$

$$\|Y\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2}$$

$$dis(X,\mathcal{B}) = \|Y\| = \frac{\sqrt{2}}{2}$$

$\boxed{31}$

$$W = \left\langle (0,1,0), \left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)\right\rangle$$

$(0,1,0)\cdot\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right) = 0$, vetores ortogonais

$$\|(0,1,0)\| = 1$$

$$\left\|\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)\right\| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

Logo, $\left\{(0,1,0), \left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)\right\}$ é uma base ortonormada de W

$$proj_W X = \left((4,0,-9)\cdot(0,1,0)\right)(0,1,0) + \left((4,0,-9)\cdot(1/2,0,\sqrt{3}/2)\right)(1/2,0,\sqrt{3}/2)$$

$$= 0\,(0,1,0) + \frac{4-9\sqrt{3}}{2}\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)$$

$$= \left(1-\frac{9\sqrt{3}}{4}, 0, \sqrt{3}-\frac{27}{4}\right)$$

$$proj_W Y = \left((2,7,-1)\cdot(0,1,0)\right)(0,1,0) + \left((2,7,-1)\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)\right)\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right)$$
$$= (0,7,0) + \left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2},0,\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}-\frac{\sqrt{3}}{4}, 7, \frac{\sqrt{3}}{4}-\frac{3}{4}\right)$$

$\boxed{3 \quad 2}$

a) Verdadeiro, se $a \neq 0$ forma uma reta com direção $(a, 0, -a)$, $a \in \mathbb{R} \setminus \{0\}$
se $a = 0 \Rightarrow$ o subespaço é o vetor nulo

b) Falso, $\{(1,1,1), (0,0,0)\}$ por exemplo.

c) Falso, pelas linhas não nulas de A_r

d) Verdadeiro, pois $\dim \mathcal{C}(A) = \dim \mathcal{L}(A) = n$, logo as linhas de A formam uma base de $\mathbb{R}^n$

e) Falso, $car(A) = 8$

f) Falso, todo o conjunto de 5 vetores l.i. em $\mathbb{R}^5$ é uma base em $\mathbb{R}^5$

g) Verdadeiro, 5 vetores ortonormados $\Rightarrow$ 5 vetores l.i. $\Rightarrow$ explicação de f)

h) Falso, por exemplo $\{(0,0,1), (0,1,0)\}$

i) Falso, por exemplo $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ é simétrica e $car(A) = 1$

j) Verdadeiro, sendo que o conjunto de vetores pode ser reduzido a 3 vetores l.i., que formam uma base de $\mathbb{R}^3$