Cálculo I - agr. 4   2020/21

2º teste - turmas TP4A-2, TP4A-5

Resolução

1. (a) (30 pontos)

$$\int (x+2)^2 \sin x \, dx = (x+2)^2 \cdot (-\cos x) - \int (-\cos x) \cdot 2(x+2) \, dx$$

$$= -(x+2)^2 \cos x + 2 \int \cos x \, (x+2) \, dx$$

$$= -(x+2)^2 \cos x + 2(x+2)\sin x - 2\int \sin x \cdot 1 \, dx$$

$$= -(x+2)^2 \cos x + 2(x+2)\sin x + 2\cos x + C, \quad C \in \mathbb{R}.$$

(b) (30 pontos)

$$\int \frac{x^2}{x^2+2x+1} \, dx$$

$$= \int \left(1 - \frac{2x+1}{(x+1)^2}\right) dx$$

$$= x - \int \frac{2x+1}{(x+1)^2} \, dx$$

$$= x - \int \left(\frac{2}{x+1} - \frac{1}{(x+1)^2}\right) dx$$

$$= x - 2\ln|x+1| - \frac{1}{x+1} + C,$$

C constante em intervalos.

C.A.:

$$\frac{x^2}{x^2+2x+1} = \frac{x^2+2x+1-(2x+1)}{x^2+2x+1}$$

$$= 1 - \frac{2x+1}{(x+1)^2}$$

$$\frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow$$

$$\Rightarrow \frac{2x+1}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

$$\Rightarrow 2x+1 = Ax + A + B$$

$$A = 2, \quad B = -1$$

(c) (40 pontos)

$$\int \frac{\sqrt{x+4}}{x} \, dx$$

$$= \int \frac{t}{t^2-4} \cdot 2t \, dt = 2\int \frac{t^2}{t^2-4} \, dt$$

$$= 2\int \frac{t^2-4+4}{t^2-4} \, dt = 2\int \left(1 + \frac{4}{t^2-4}\right) dt$$

$$= 2\int \left(1 + \frac{1}{t-2} - \frac{1}{t+2}\right) dt$$

C.A.: Mudança de variável

$$\sqrt{x+4} = t \Leftrightarrow x = t^2-4, \quad t > 0$$

$$\frac{dx}{dt} = 2t > 0 \quad \text{(sinal constante)}$$

C.A.

$$\frac{4}{t^2-4} = \frac{A}{t-2} + \frac{B}{t+2}$$

$$\frac{4}{t^2-4} = \frac{A(t+2)+B(t-2)}{(t-2)(t+2)}$$

$$4 = At + 2A + Bt - 2B$$

$$\begin{cases} A + B = 0 \\ 2A - 2B = 4 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = -1 \end{cases}$$

$= 2t + 2\ln|t-2| - 2\ln|t+2| + c$

$= 2\sqrt{x+4} + 2\ln|\sqrt{x+4}-2| - 2\ln(\sqrt{x+4}+2) + c,$

c constante em intervalos.

---

2. $A = \{(x,y)\in\mathbb{R}^2 : x \geq 0 \wedge \frac{x+2}{3} \leq y \leq \sqrt{x}\}$

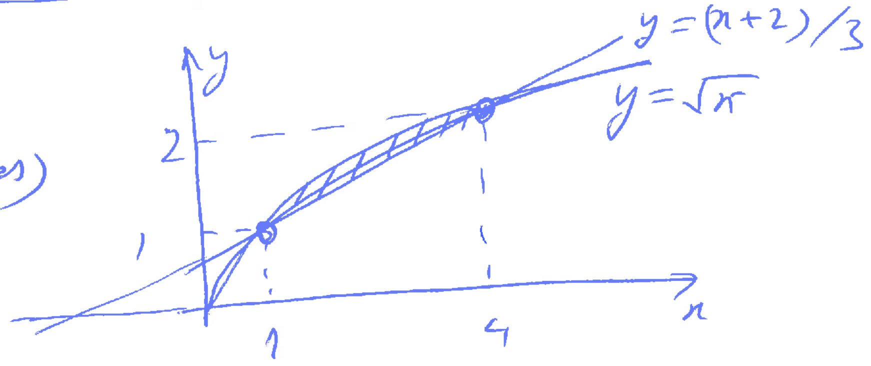
(a)
(20 pontos)

$$\begin{cases} y = \frac{x+2}{3} \\ y = \sqrt{x} \\ x \geq 0 \end{cases} \Leftrightarrow \begin{cases} \frac{x+2}{3} = \sqrt{x} \\ = \\ = \end{cases} \Leftrightarrow \begin{cases} (x+2)^2 = 9x \\ = \\ = \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2 - 5x + 4 = 0 \\ = \\ = \end{cases} \Leftrightarrow \begin{cases} x = \frac{5 \pm 3}{2} \\ = \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \vee \begin{cases} x = 4 \\ y = 2 \end{cases}$$

Os pontos de interseção são $(1,1)$ e $(4,2)$.

(b)
(20 pontos)

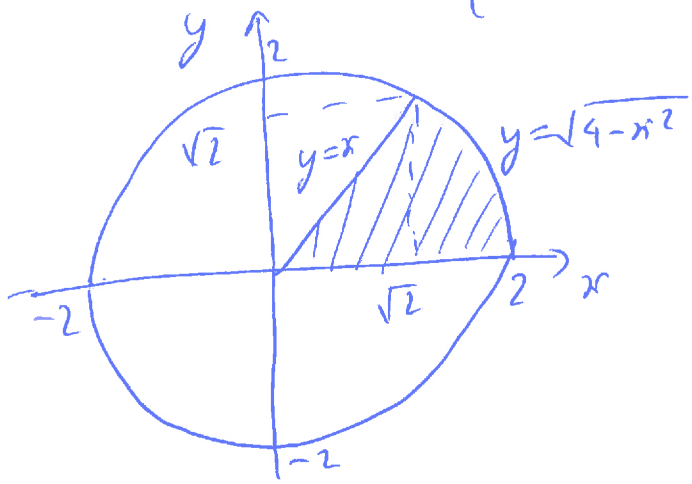


(c)
(30 pontos)

$$\int_1^4 \left(\sqrt{x} - \frac{x+2}{3}\right)dx = \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{6} - \frac{2x}{3}\right]_1^4$$

$$= \frac{2}{3}\cdot 8 - \frac{16}{6} - \frac{8}{3} - \frac{2}{3} + \frac{1}{6} + \frac{2}{3} = \frac{1}{6}$$

3. (30 pontos) $g(x) = \begin{cases} x, & \text{se } 0 \leq x \leq \sqrt{2} \\ \sqrt{4-x^2}, & \text{se } \sqrt{2} < x \leq 2 \end{cases}$.



C.A.: $y=\sqrt{4-x^2}$

$\Rightarrow y^2 = 4-x^2 \wedge y \geq 0$

$\Leftrightarrow x^2+y^2=4 \wedge y \geq 0$

$\Rightarrow$ circunferência de raio $R=2$

$$\int_0^2 g(x)\,dx = \text{Área da zona sombreada} = \frac{1}{8}\cdot \pi R^2$$
$$= \frac{1}{8}\cdot 4\pi = \frac{\pi}{2}.$$

Cálculo direto: $\int_0^2 g(x)\,dx = \int_0^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^2 \sqrt{4-x^2}\,dx$

$$= \left[\frac{x^2}{2}\right]_0^{\sqrt{2}} + \int_{\pi/4}^{\pi/2} 2\cos t \cdot 2\cos t\,dt$$

C.A.: $x = 2\sin t$, $t \in [-\pi/2, \pi/2]$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2\cos t$

$dx = 2\cos t\,dt$

$t = \arcsin \frac{x}{2}$

x	$\sqrt{2}$	2
t	$\frac{\pi}{4}$	$\frac{\pi}{2}$

$$= \frac{\sqrt{2}^2}{2} - 0 + \int_{\pi/4}^{\pi/2} 4\cos^2 t\,dt$$

$$= 1 + \int_{\pi/4}^{\pi/2} 2(1+\cos 2t)\,dt$$

$$= 1 + \left[2t + \sin(2t)\right]_{\pi/4}^{\pi/2}$$

$$= 1 + \pi + \sin\pi - \frac{\pi}{2} - \sin\frac{\pi}{2} = 1 + \pi + 0 - \frac{\pi}{2} - 1 = \frac{\pi}{2}.$$