Universidade de Aveiro
Departamento de Matemática

Cálculo I— Agrupamento 4 2019/2020

Ficha de Exercícios 2

Primitivação (Integração indefinida)

Exercícios propostos

1. Calcule os seguintes integrais indefinidos:

(a) $\displaystyle\int (3x^2+5x+7)\,dx$ (b) $\displaystyle\int \sqrt[3]{x}\,dx$ (c) $\displaystyle\int (x^3+1)^2\,dx$ (d) $\displaystyle\int \frac{\operatorname{arctg} x}{1+x^2}\,dx$

(e) $\displaystyle\int \frac{3x^2}{1+x^3}\,dx$ (f) $\displaystyle\int \frac{1}{x^7}\,dx$ (g) $\displaystyle\int \frac{x+1}{2+4x^2}\,dx$ (h) $\displaystyle\int 4x^3\cos x^4\,dx$

(i) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ (j) $\displaystyle\int \operatorname{sen} x\cos^5 x\,dx$ (k) $\displaystyle\int \operatorname{tg} x\,dx$ (l) $\displaystyle\int \frac{\ln x}{x}\,dx$

(m) $\displaystyle\int e^{\operatorname{tg} x} sec^2 x\,dx$ (n) $\displaystyle\int x7^{x^2}\,dx$ (o) $\displaystyle\int \operatorname{sen}(\sqrt{2}x)\,dx$ (p) $\displaystyle\int \frac{x^2+1}{x}\,dx$

(q) $\displaystyle\int \frac{x}{(7+5x^2)^{\frac{3}{2}}}\,dx$ (r) $\displaystyle\int \frac{x^3}{1+x^8}\,dx$ (s) $\displaystyle\int \frac{5x^2}{\sqrt{1-x^6}}\,dx$ (t) $\displaystyle\int \frac{1}{x^2+7}\,dx$

2. Determine a primitiva F para a função $f(x)=\frac{2}{x}+\frac{3}{x^2}$, no intervalo $]-\infty,0[$, tal que $F(-1)=1$.

3. Sabendo que a função f satisfaz a igualdade $\displaystyle\int f(x)\,dx = \operatorname{sen} x - x\cos x - \frac{1}{2}x^2 + c$, com $c\in\mathbb{R}$, determinar $f(\frac{\pi}{4})$.

4. Seja f a função de domínio $\mathbb{R}^+$ tal que $f(x)=\dfrac{1}{x^2}+1$. Determine a primitiva de f que se anula em $x=2$.

5. Determine a função g que verifica as seguintes condições:

$$g'(x)=\frac{1}{(1+\operatorname{arctg}^2(x))(1+x^2)} \quad \text{e} \quad \lim_{x\to+\infty} g(x)=0.$$

6. Calcule, usando a técnica de integração por partes, os seguintes integrais indefinidos:

(a) $\displaystyle\int x\cos x\,dx$ (b) $\displaystyle\int x^2\cos x\,dx$ (c) $\displaystyle\int e^{-3x}(2x+3)\,dx$ (d) $\displaystyle\int \ln^2 x\,dx$

(e) $\displaystyle\int e^{2x}\operatorname{sen}(x)\,dx$ (f) $\displaystyle\int \operatorname{sen}(\ln x)\,dx$ (g) $\displaystyle\int \operatorname{arcsen} x\,dx$ (h) $\displaystyle\int x\operatorname{arcsen} x^2\,dx$

(i) $\displaystyle\int \operatorname{arctg} x\,dx$ (j) $\displaystyle\int \operatorname{arctg}\frac{1}{x}\,dx$ (k) $\displaystyle\int \sqrt{x}\ln x\,dx$ (l) $\displaystyle\int \sin x\cos x\,dx$

7. Usando integração quase-imediata e/ou integração por partes, determine:

(a) $\displaystyle\int \operatorname{cosec} x\,dx$ (b) $\displaystyle\int \frac{\operatorname{tg}^3 x}{\cos^2 x}\,dx$ (c) $\displaystyle\int \operatorname{cotg}^2 x\,dx$ (d) $\displaystyle\int \cos^2\theta\,d\theta$

(e) $\displaystyle\int \operatorname{sen}^2 x\,dx$ (f) $\displaystyle\int \operatorname{sen}^3 t\,dt$ (g) $\displaystyle\int \operatorname{tg}^4 x\,dx$ (h) $\displaystyle\int \operatorname{sen}(3x)+\cos(5x)\,dx$

(i) $\displaystyle\int \operatorname{tg} x\sec^2 x\,dx$ (j) $\displaystyle\int \operatorname{sen}^5 x\cos^2 x\,dx$ (k) $\displaystyle\int \operatorname{sen}^2 x\cos^4 x\,dx$ (l) $\displaystyle\int \cos x\cos(5x)\,dx$

(m) $\displaystyle\int \frac{1}{x}\cos(\ln x)\,dx$ (n) $\displaystyle\int x^5\mathrm{sen}\,(x^6)\,dx$ (o) $\displaystyle\int \frac{\arccos x - x}{\sqrt{1-x^2}}\,dx$ (p) $\displaystyle\int \frac{\cos(\ln(x^2))}{x}\,dx$

8. Calcule os seguintes integrais indefinidos:

   (a) $\displaystyle\int \frac{x+2}{x^2+5x-6}\,dx$ (b) $\displaystyle\int \frac{1}{(x-1)(x+1)^3}\,dx$ (c) $\displaystyle\int \frac{1}{x^3+8}\,dx$ (d) $\displaystyle\int \frac{x^4-4x^2+3}{x^2-9}\,dx$

   (e) $\displaystyle\int \frac{x^3+3x-1}{x^4-4x^2}\,dx$ (f) $\displaystyle\int \frac{x^4}{x^4-1}\,dx$ (g) $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$ (h) $\displaystyle\int \frac{x+1}{x^2+4x+5}\,dx$

9. Calcule, usando a técnica de integração por substituição, os seguintes integrais indefinidos:

   (a) $\displaystyle\int x^2\sqrt{1-x}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$ (c) $\displaystyle\int x(2x+5)^{10}\,dx$ (d) $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$

   (e) $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx$ (f) $\displaystyle\int \frac{1}{x\sqrt{x^2+4}}\,dx$ (g) $\displaystyle\int \sqrt{3-2x^2}\,dx$ (h) $\displaystyle\int \frac{x^2}{\sqrt{1-2x-x^2}}\,dx$

   (i) $\displaystyle\int \frac{1}{x^2\sqrt{x^2-7}}\,dx$ (j) $\displaystyle\int \frac{1}{\sqrt{2x+3}+\sqrt[3]{(2x+3)^2}}\,dx$ (k) $\displaystyle\int e^{\sqrt{x}}\,dx$ (l) $\displaystyle\int \frac{\ln x}{x\cdot\sqrt{1+\ln x}}\,dx$

10. Calcule

    (a) $\displaystyle\int \frac{x+1}{\sqrt{3-x^2}}\,dx$ (b) $\displaystyle\int \mathrm{sen}^{\,4}x\,dx$ (c) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$ (d) $\displaystyle\int \frac{1}{\sqrt{2+x^2}}\,dx$

    (e) $\displaystyle\int \frac{\mathrm{sen}\,\sqrt{x}}{\sqrt{x}}\,dx$ (f) $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$ (g) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$ (h) $\displaystyle\int x\sqrt{(1+x^2)^3}\,dx$

    (i) $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$ (j) $\displaystyle\int x\ln x\,dx$ (k) $\displaystyle\int \frac{1+e^x}{e^{2x}+4}\,dx$ (l) $\displaystyle\int x\mathrm{arctg}\,x\,dx$

    (m) $\displaystyle\int \frac{\mathrm{sen}\,x}{(1-\cos x)^3}\,dx$ (n) $\displaystyle\int (2x^2+3)\mathrm{arctg}\,x\,dx$ (o) $\displaystyle\int \frac{1}{\sqrt{x^2+2x-3}}\,dx$ (p) $\displaystyle\int \sqrt{1+e^x}\,dx$

    (q) $\displaystyle\int \frac{1}{\sqrt{e^x-1}}\,dx$ (r) $\displaystyle\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$ (s) $\displaystyle\int \frac{\ln x}{x(\ln^2 x+1)}\,dx$ (t) $\displaystyle\int x^3e^{x^2}\,dx$

    (u) $\displaystyle\int \frac{2x-1}{(x-2)(x-3)(x+1)}\,dx$ (v) $\displaystyle\int \frac{1+\mathrm{tg}^{\,2}x}{\sqrt{\mathrm{tg}\,x-1}}\,dx$ (w) $\displaystyle\int \frac{x^8}{1+x^2}\,dx$ (x) $\displaystyle\int \frac{x+1}{x^3-1}\,dx$

11. Usando a substituição $x = 2\mathrm{tg}\,t$, $t\in]0,\frac{\pi}{2}[$, calcule $\displaystyle\int \frac{3x+7}{(x^2+4)^2}\,dx$.

12. [1] Calcule os seguintes integrais indefinidos:

    (a) $\displaystyle\int \frac{x^2}{\sqrt{1+x^3}}\,dx$ (b) $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$ (c) $\displaystyle\int \frac{3x-1}{x^3+x}\,dx$

    (d) $\displaystyle\int \frac{1}{e^{2x}+2}\,dx$, usando a mudança de variável $e^x = t$.

    (e) $\displaystyle\int \frac{-\cos x}{(1+\mathrm{sen}\,x)^2}\,dx$ (f) $\displaystyle\int x\cdot\ln(1+x^2)\,dx$ (g) $\displaystyle\int \cos x\cdot\ln(\mathrm{sen}\,x))\,dx$

13. Determine a função $f:\mathbb{R}\to\mathbb{R}$ tal que $f(0)=1, f'(0)=2$ e $f''(x)=12x$, para todo o $x\in\mathbb{R}$.

14. Determine a função $f:\mathbb{R}\to\mathbb{R}$ tal que $f'(x)=\dfrac{2e^x}{3+e^x}$ e $f(0)=\ln 4$.

15. Determine a função f, de domínio $\mathbb{R}^+$, tal que $f'(x)=\dfrac{5x-4}{x(x^2-2x+2)}$ e $\displaystyle\lim_{x\to+\infty} f(x)=0$.

[1] A partir daqui os exercícios propostos foram retirados de provas de avaliação de anos anteriores.

Exercícios Resolvidos

1. Considere a função g definida em $\mathbb{R}^+$ por $g(x) = \dfrac{(\ln x)^2}{x}$.

   (a) Determine a família de todas as primitivas de g.

   (b) Indique a primitiva da função g que se anula para $x = e$.

   Resolução:

   (a) $\displaystyle\int \frac{(\ln x)^2}{x}\, dx = \frac{(\ln x)^3}{3} + c, \quad c \in \mathbb{R}.$

   (b) Para cada $c \in \mathbb{R}$, $G(x) = \frac{(\ln x)^3}{3} + c$ é uma primitiva de g.
   Pretendemos então determinar $c \in \mathbb{R}$ tal que $G(e) = 0$.

   $$G(e) = 0 \Leftrightarrow \frac{1}{3} + c = 0 \Leftrightarrow c = -\frac{1}{3}$$

   Assim, $G(x) = \frac{(\ln x)^3}{3} - \frac{1}{3}$ é a primitiva de g que se anula para $x = e$.

2. Calcule $\displaystyle\int (x+1)\operatorname{sen} x\, dx$, usando primitivação por partes.

   Resolução: Fazendo

   $$\begin{array}{c} f'(x) = \operatorname{sen} x \quad \text{temos} \quad f(x) = -\cos x \\ g(x) = x + 1 \quad \text{temos} \quad g'(x) = 1 \end{array}$$

   Assim,

   $$\begin{aligned} \int (x+1)\operatorname{sen} x\, dx &= -(x+1)\cos x + \int \cos x\, dx \\ &= -(x+1)\cos x + \operatorname{sen} x + c, \quad c \in \mathbb{R} \end{aligned}$$

3. Calcule $\displaystyle\int x\sqrt{x+1}\, dx$

   Resolução:

   Consideremos a substituição $x + 1 = t^2$, com $t \geq 0$. Definindo $\varphi(t) = t^2 - 1$, $t \geq 0$, temos que φ é invertível, diferenciável e $\varphi'(t) = 2t$. Então

   $$\begin{aligned} \int x\sqrt{x+1}\, dx &= \int (t^2 - 1)\cdot t \cdot 2t\, dt \\ &= \frac{2t^5}{5} - \frac{2t^3}{3} + c\,. \end{aligned}$$

   Atendendo a que $x + 1 = t^2$, com $t \geq 0$, vem que $t = \sqrt{x+1}$. Assim,

   $$\int x\sqrt{x+1}\, dx = \frac{2(x+1)^2\sqrt{x+1}}{5} - \frac{2(x+1)\sqrt{x+1}}{3} + c, \text{ com } c \in \mathbb{R}.$$

4. Calcule $\displaystyle\int \frac{x+2}{(x-1)^2(x^2+4)}\,dx$

Resolução:

O cálculo deste integral indefinido passa por decompor em frações simples a fração

$$\frac{x+2}{(x-1)^2(x^2+4)}$$

Isto é, passa por escrever a dita fração na seguinte forma

$$\frac{x+2}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} \quad (*)$$

com $A,\ B,\ C$ e D constantes reais a determinar.

Temos então que

$$\frac{x+2}{(x-1)^2(x^2+4)} = \frac{(A+C)x^3 + (-A+B-2C+D)x^2 + (4A+C-2D)x - 4A + 4B + D}{(x-1)^2(x^2+4)}$$

donde resulta a igualdade de polinómios

$$x+2 = (A+C)x^3 + (-A+B-2C+D)x^2 + (4A+C-2D)x - 4A + 4B + D.$$

Atendendo à condição de igualdade de polinómios resulta que

$$\begin{cases} A+C=0 \\ -A+B-2C+D=0 \\ 4A+C-2D=1 \\ -4A+4B+D=2 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{1}{25} \\ B=\frac{15}{25} \\ C=\frac{1}{25} \\ D=-\frac{14}{25} \end{cases}$$

Voltando a $(*)$, podemos escrever

$$\frac{x+2}{(x-1)^2(x^2+4)} = \frac{-\frac{1}{25}}{x-1} + \frac{\frac{15}{25}}{(x-1)^2} + \frac{\frac{1}{25}x - \frac{14}{25}}{x^2+4}$$

Assim

$$\begin{aligned}
\int \frac{x+2}{(x-1)^2(x^2+4)}\,dx &= -\tfrac{1}{25}\int \frac{1}{x-1}\,dx + \frac{15}{25}\int (x-1)^{-2}\,dx + \frac{1}{25}\int \frac{x-14}{x^2+4}\,dx \\
&= -\tfrac{1}{25}\ln|x-1| - \tfrac{3}{5(x-1)} + \tfrac{1}{25}\int \frac{x}{x^2+4}\,dx - \frac{14}{25}\int \frac{1}{x^2+4}\,dx \\
&= -\tfrac{1}{25}\ln|x-1| - \tfrac{3}{5(x-1)} + \tfrac{1}{50}\int \frac{2x}{x^2+4}\,dx - \frac{7}{25}\int \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}\,dx \\
&= -\tfrac{1}{25}\ln|x-1| - \tfrac{3}{5(x-1)} + \tfrac{1}{50}\ln(x^2+4) - \tfrac{7}{25}\operatorname{arctg}\tfrac{x}{2} + c, \quad c \in \mathbb{R}
\end{aligned}$$